PROGRAM ROZWOJOWY
POLITECHNIKI WARSZAWSKIEJ

Course Title: FINITE ELEMENT METHOD I
Type of course: undergraduate, graduate

Field of study (Programme), specialization
Aerospace Engineering, Power Engineering, Robotics, Computer Aided Engineering

Format (Teaching methods)	Lectures: 2hours/week	Laboratory: 1hour/week	Private study: 2hours/week

Lecturers (course leader): Grzegorz Krzesiński

Objective: To supply the basic knowledge and skills required for understanding and simple practical applications of FEM

Contents (lecture's programme):
1. INTRODUCTION TO FINITE ELEMENT METHOD
2. APPLICATIONS OF FEM
3. FINITE DIFFERENCE METHOD (FDM), BOUNDARY ELEMENT METHOD (BEM) AND FINITE ELEMENT METHOD (FEM)
4. BEAMS. RITZ-RAYLAYGH METHOD and FINITE ELEMENT METHOD
5. BARS AND SPRINGS
6. TRUSSES AND FRAMES
7. TWO AND THREE- DIMENSIONAL LINEAR ELASTOSTATICS
8. CST TRIANGULAR ELEMENT
9. 8-NODE QUADRILATERAL ELEMENT . NUMERICAL INTEGRATION

Computer lab: Introduction to practical problems of FE modeling in ANSYS/ 2D and 3D linear stress analysis/ Static analysis of simple shell structure/ Discretization error and adaptive meshing

Abilities: After completing the course the students will be able to build simple FE models and will know the possible applications and limitations of the method in mechanics of structures.

Assesment method: Assesment based on tests and results of computer lab work (reports).
Practical work: Project/laboratory classes, where students will build and analyse the results of simple FE models of elastic structures

Recommended texts (reading):

[1] Huebner K. H., Dewhirst D. L., Smith D.E., Byrom T. G.: The finite element method for engineers, J. Wiley & Sons, Inc., 2001.

[2] Zagrajek T., Krzesinski G., Marek P.: MES w mechanice konstrukcji. Ćwiczenia z zastosowaniem programu ANSYS, Of.Wyd.PW 2005

[3] Bijak-Żochowski M., Jaworski A., Krzesiński G., Zagrajek T.: Mechanika Materiałów i Konstrukcji, Tom 2, Warszawa, Of. Wyd. PW, 2005

[4] Saeed Moaveni: Finite Element Analysis. Theory and Application with ANSYS, Paerson Ed. 2003

[5] Cook R. D.: Finite Element Modeling for Stress Analysis, John Wiley & Sons , 1995

[6] Zienkiewicz O.C., Taylor R.: The Finite Element Method.- different publishers and editions

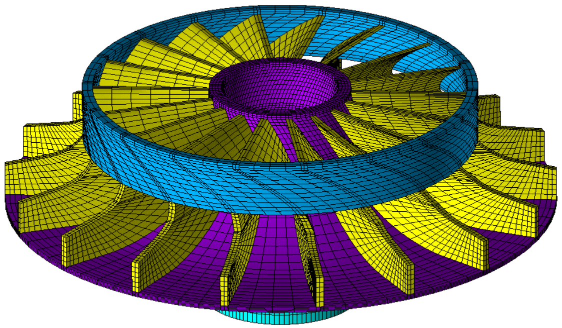
1. INTRODUCTION TO FINITE ELEMENT METHOD

The FEM is a numerical procedure that can be used to solve a large class of engineering problems including mechanics of structures, heat transfer, electromagnetism, fluid flow and coupled fields problems (e.g. electro-thermal).

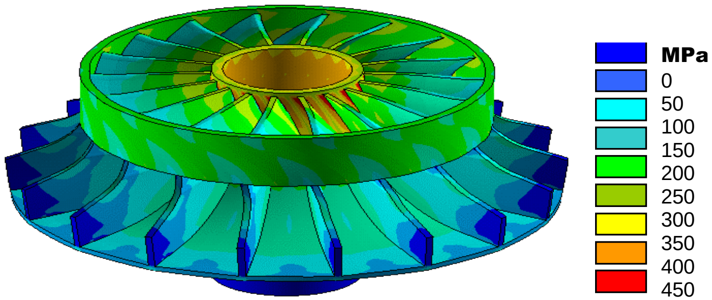
The simplest description.

The method involves dividing the geometrical model of the analysed structure into very small, simple pieces called **finite elements,** connected by nodes. The behaviour of of the element is described by adequate physical laws. An unknown quantity (e.g. temperature, displacement vector, electrical potential) is interpolated over an element from the nodal values using specially defined polynomials (called shape functions). The procedure leads to the set of simultaneus algebraic equations with the nodal values being unknown.

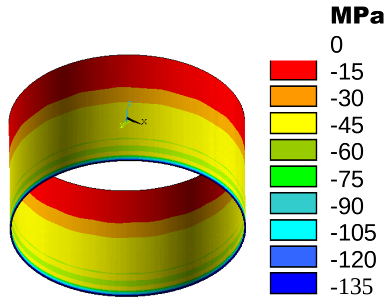
During the solution process the nodal values (DOF- degrees of freedom of the model) are found. Then all interesting quantities (strains, stresses) are calculated within the elements. Finally the results may be presented in the required graphical form (the typical form of presentation is a contour map)



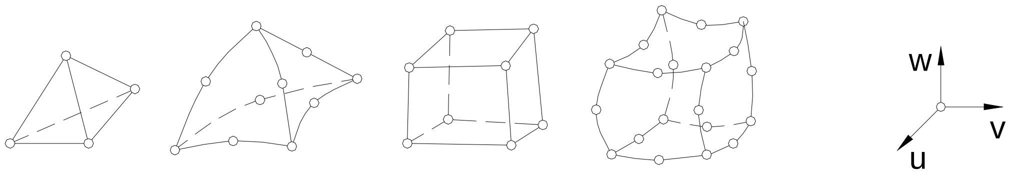
Finite Element Mesh



Von Mises stress distribution



Contact pressure between the shaft and the rotor disk



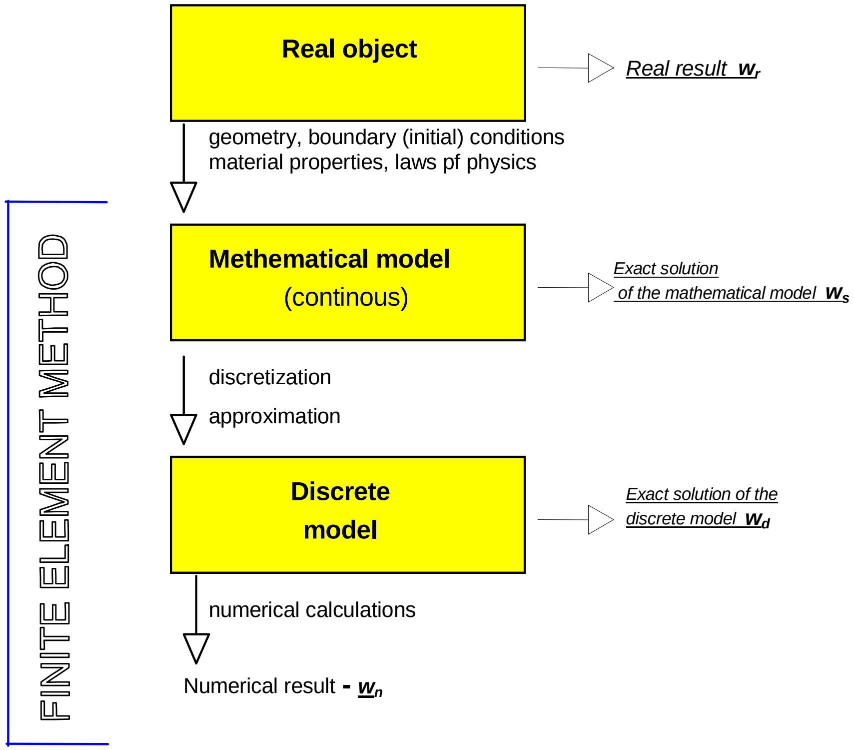
3D finite elements

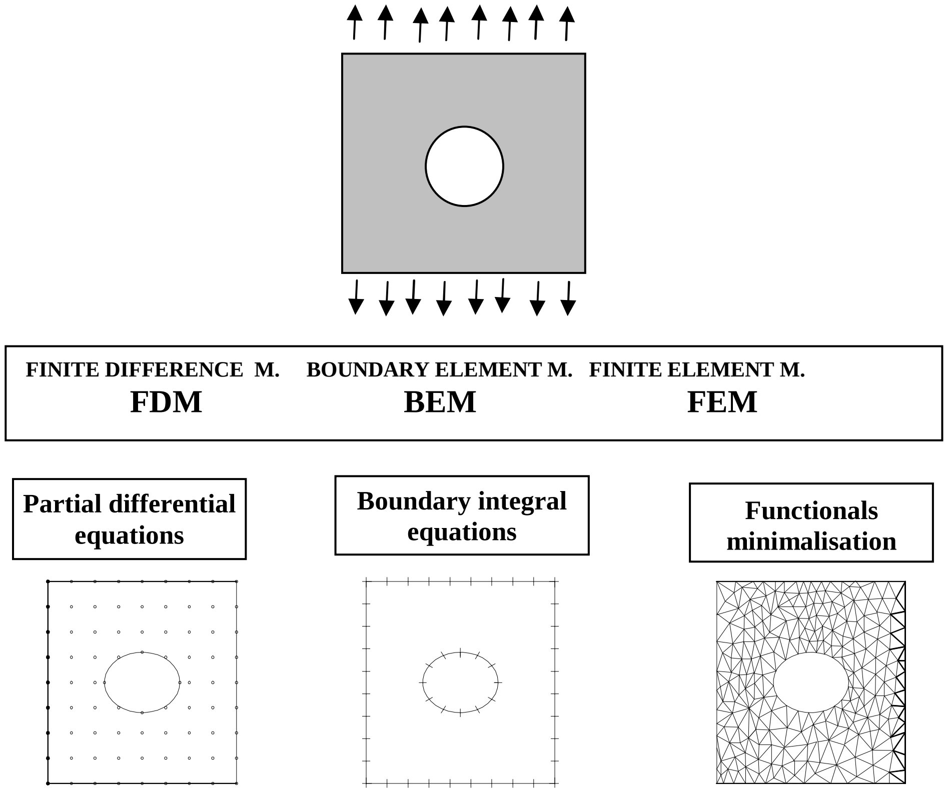
FEM was developed in 1950's for solving complex problems of stress analysis - mainly for aeronautical industry. The development of the method was connected with the progress in digital computers and numerical techniques.

Today the method is considered as the most powerful analysis method for problems described by partial differential equations.

FEM is one of the approximate methods for solving continuous problems of mathematics and physics

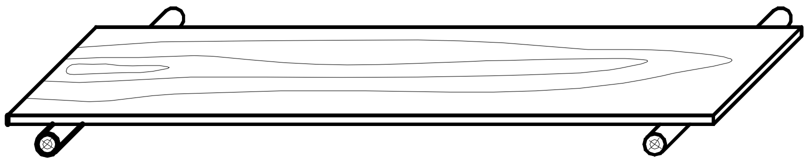
Approximate methods – flow chart



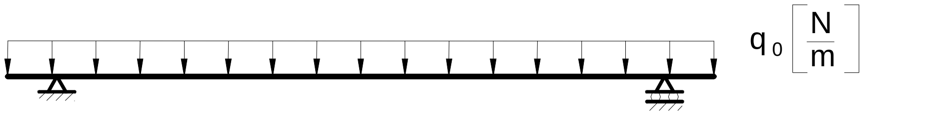


Discretization of the continuous problem – numerical estimation of the integral

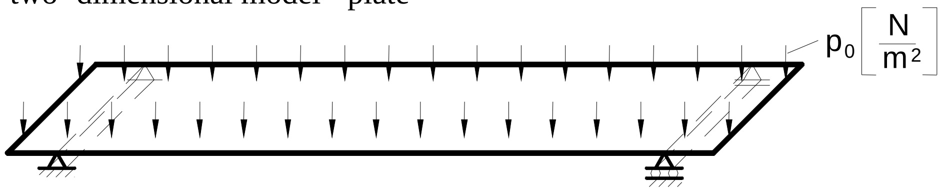
The example – wooden board
Different models for the problem



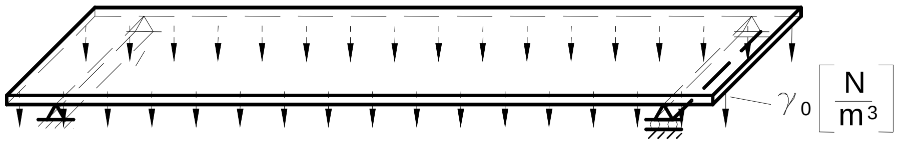
one dimensional model - beam

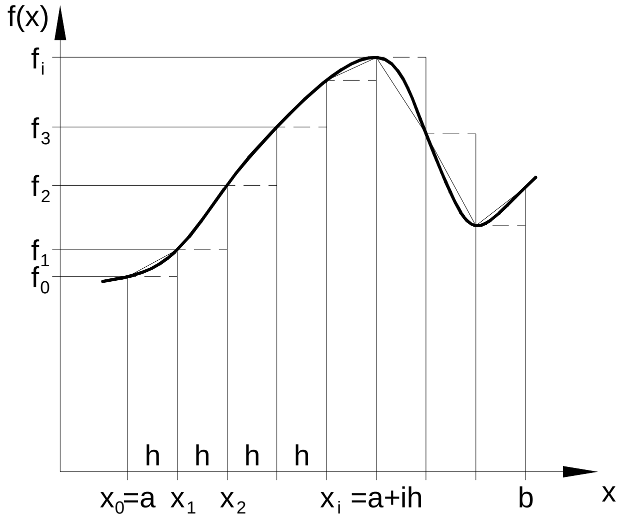


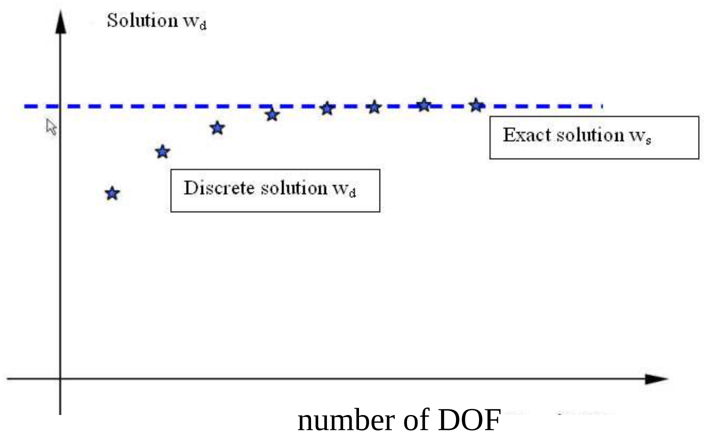
two- dimensional model - plate



c) three dimensional model – solid volume







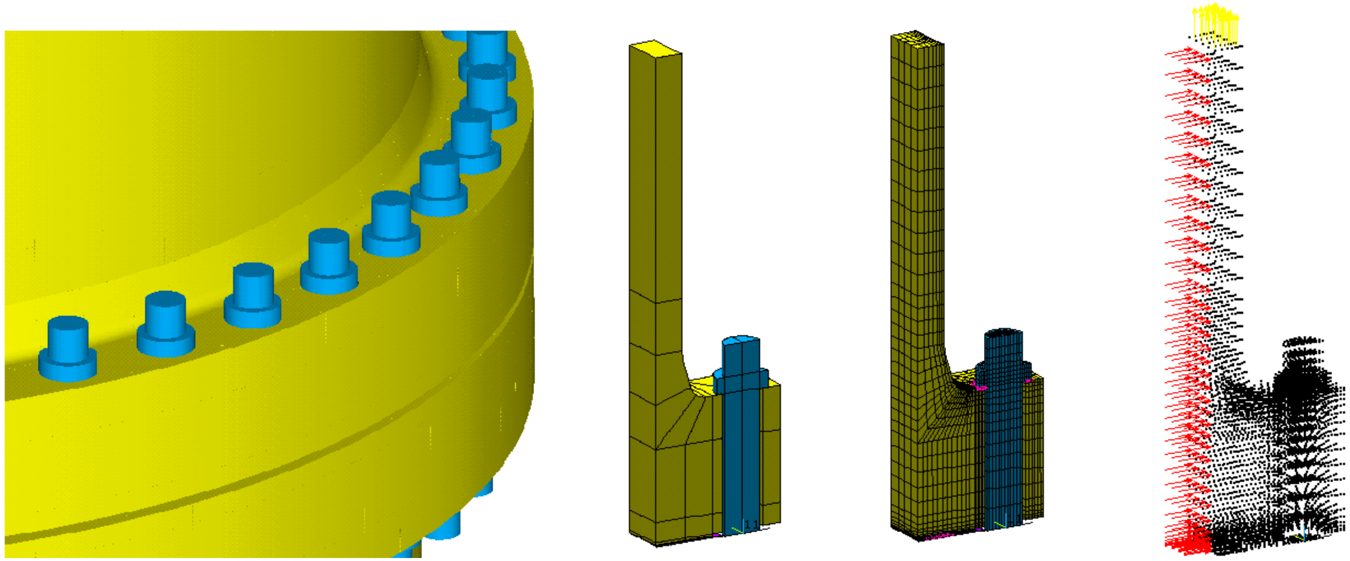
BASIC STEPS IN THE FINITE ELEMENT METHOD (FE modeling)

Preprocessor (preprocessing phase)

In the preprocessing phase the mathematical problem is described and presented in the numerical, discrete form:

Steps:
Description of :

the analysed domain (geometry of the analysis object)
the material properties
the boundary conditions (loads and constraints)
the meshing (dividing the domain into the finite elements of the required density distribution)



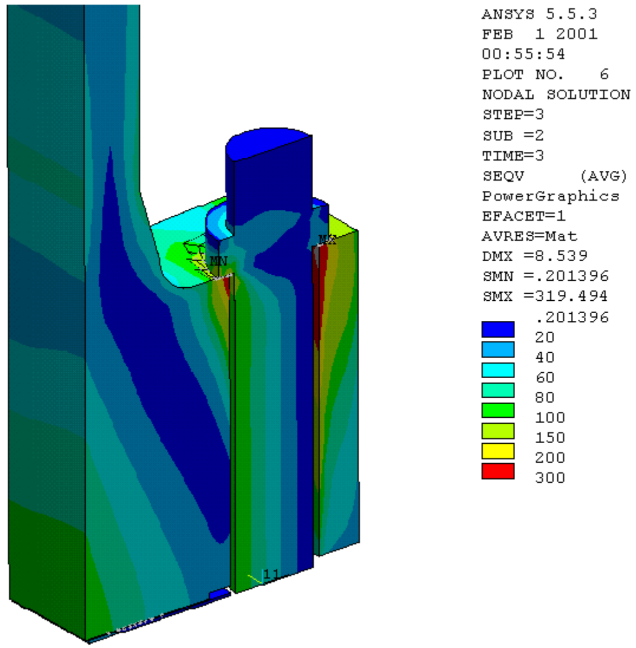
FE model of the bolted joint of the high pressure vessel
entire connection, representative part of the structure and its discretization, FE nodes with load symbols

Processor (solution phase)

In this phase the user of the FE program defines the type of analysis (static, linear or nonlinear, dynamic, buckling ......) and other details describing the method of calculations and solution process.
The FE program performs the calculations and writes the results in the adequate files.

Postprocesor

In this phase it is possible to present the interesting results in different forms: plots, graphs, animations, listings etc.
The user can create contour maps, tables, graphs and generate the reports.

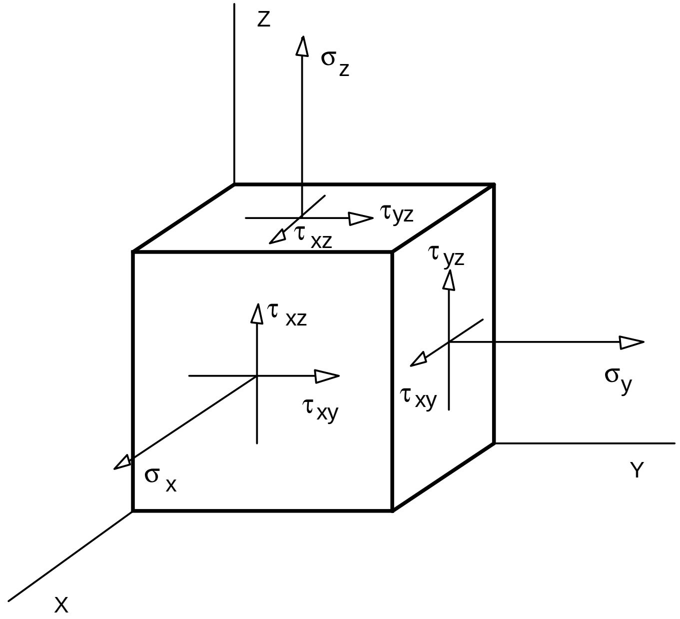


Von Mises equivalent stress distribution (MPa). Contour map

The results of FE analysis

Deformed model compared to undeformed structure

Displacement vector (u_x, u_y , u_z)
Stress state components within the model
{σ_x , σ_y , σ_z ,τ_{xy} , τ_{yz} ,τ_{xz} }



Strain state components
{ε_x , ε_y , ε_z , γ_{xy} , γ_{yz} , γ_{xz} }
Principal stresses .

Equivalent stress distribution according to an arbitrary criterion
Any other entity defined by the user (ANSYS – APDL commands)

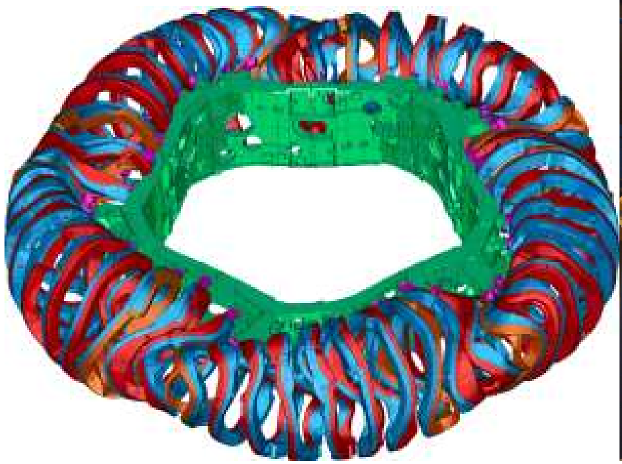
2. APPLICATIONS OF FEM

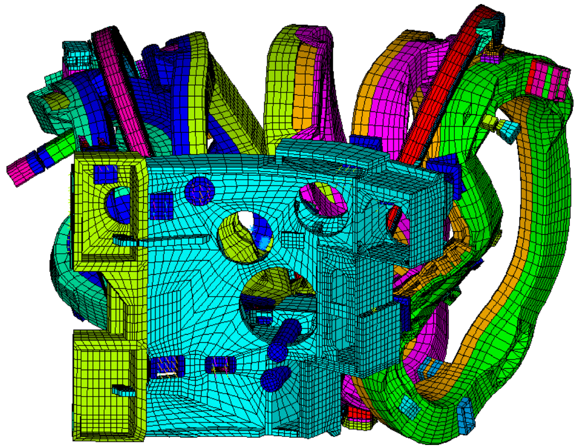
Finite Element Analysis of Critical Central Connection Elements of W7-X Stellator Coil Support System

The objective of Wendelstein 7-X project is the stellator-type fusion reactor. In this device plasma channel is under control of magnetic field coming from magnet system of complex shape, made of 70 superconducting coils symmetrically arranged in 5 identical sections. Every coil is connected to central ring with two extensions which transfer loads resulting from electromagnetic field and gravity.

During operation at a service temperature (ST) of 4K the superconducting coils of the W7-X magnet system exert high electromagnetic loads. Therefore, the detailed analysis of the coil - central support connections, the so called Central Support Elements (CSE), is a critical item for W7-X. Each coil is fastened to the CSS by two central support elements (CSE).

The aim of this work was to analyse mechanical behaviour of the bolted connections using detailed 3D finite element models (including bolts , washers, welds etc). The Global Model of the structure, analysed by Efremov Institute in Russia, provided information about the loads acting on the connections.

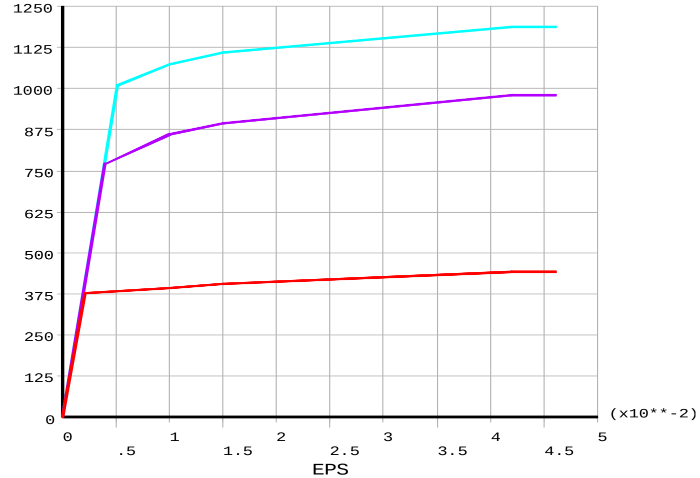




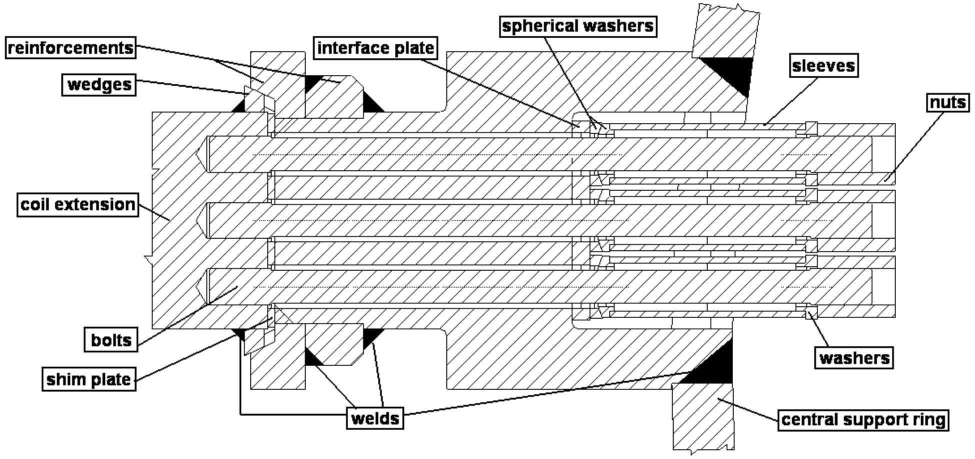
Design analyses of the support structure elements: nonlinear simulations including contact with friction, plasticity,assembly stresses, submodelling technique and using parametric models (14 bolted connections) . The work performed for Institute of Plasma Physics, Greifswald , Germany.

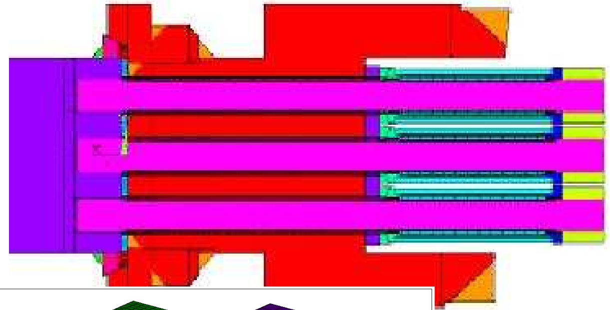
The results of the numerical simulation help to check the magnitudes of displacements and stresses for different loading scenarios and some modifications of the considered structures.

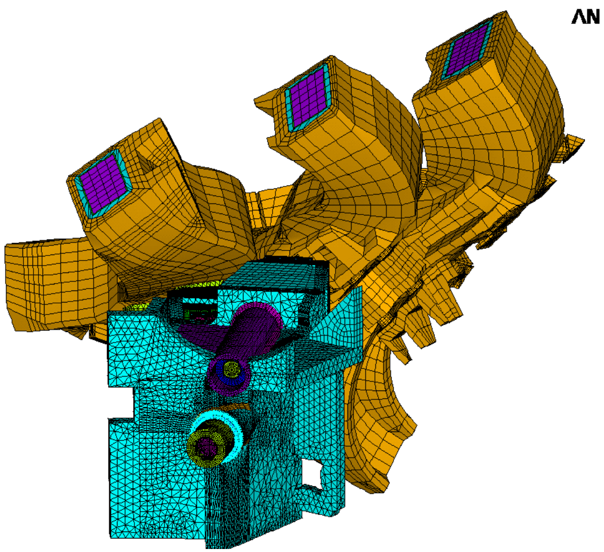
material 1 --- 1.4429 steel: shim, wedges, ring,

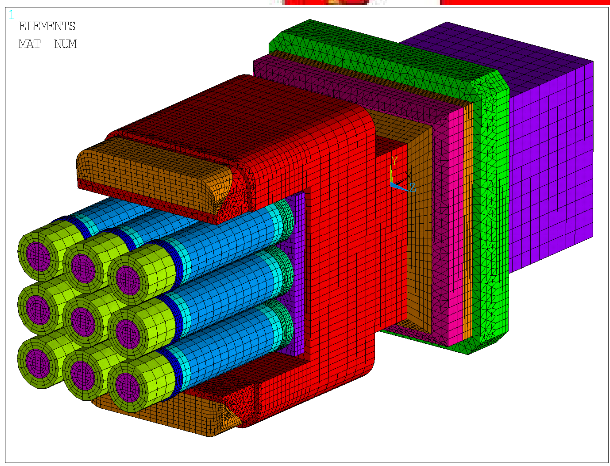


Stress-strain curve for material 1 (1.4429 steel) corresponding to the temperature 293 K (red graph), 77K (violet), 4K (blue)





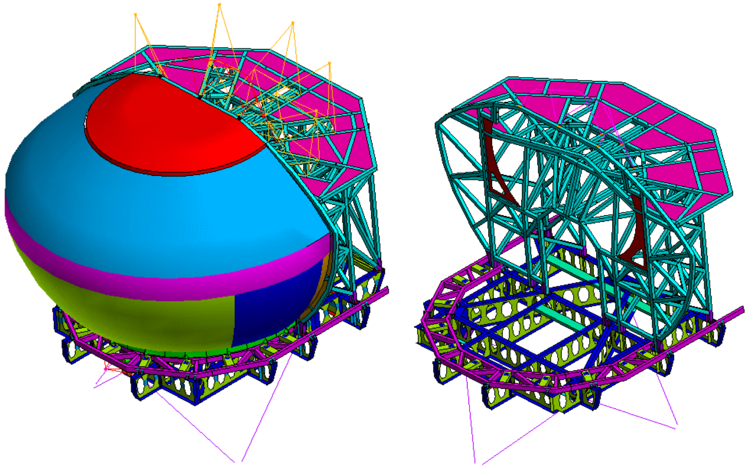




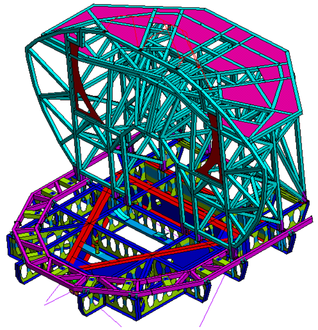
Structural analysis and design of the "KLAUDIA" flight simulator

The FE model of the initial platform design has showed the structure to be too flexibile. To find better solution the simplified FE model has been built, easy for modifications. The model has enabled quick verification of new concepts. The final detailed FE model has confirmed the improvement of the design. The fully nonlinear FE submodels have been built to check the stress level in the main joints. Vibration characteristics (natural frequencies and mode shapes) of the structure have been found

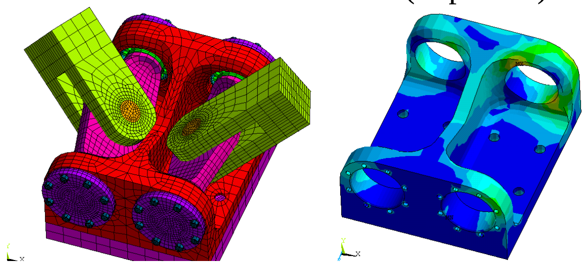
The FE model was built using shell, solid, beam, mass and link elements. The project was done for MP-PZL Aerospace Industries , Poland



Initial FE model



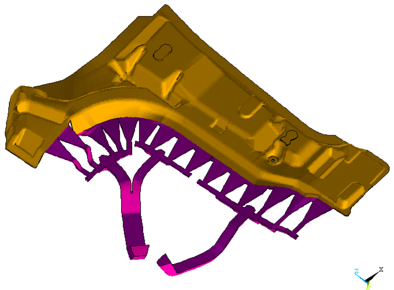
Modified (improved) design



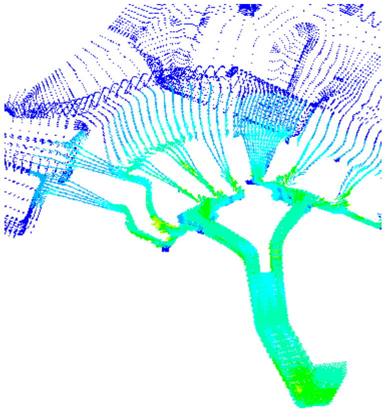
submodel of the joint

FE analysis of thin-walled elements' deformation during aluminium injection moulding

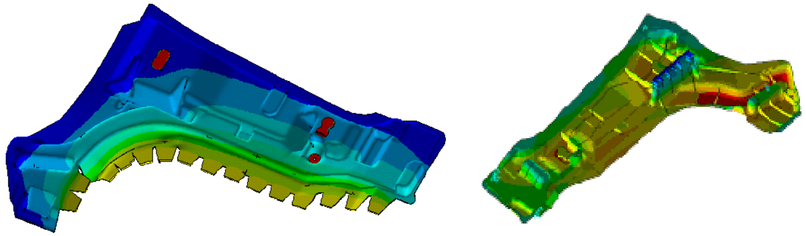
Numerical simulations have been performed to model the process of filling the mould by hot aluminium alloy. The analysis has enabled improvements of the element stiffness diminishing geometrical changes caused by the process. Fluid flow simulation with transient thermal analysis including phase change have been performed, followed by the structural elasto-plastic calculation of residual effects. The project performed for Alusuisse Technological Center, Sierre, Switzerland.



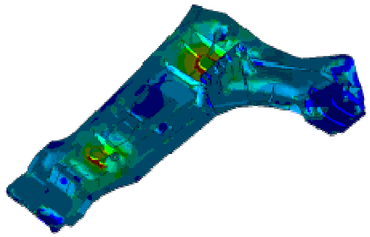
FE model



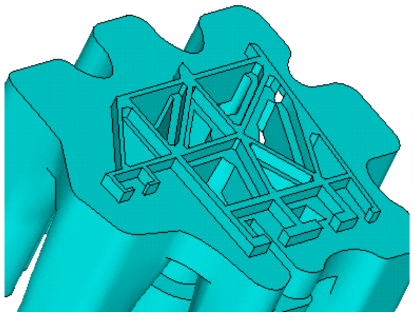
Velocity field during injection



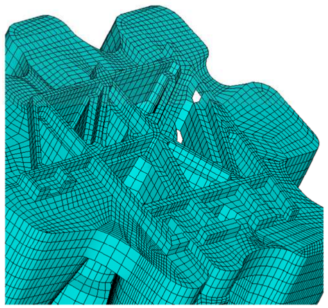
Temperature distribution (cooling effect) and displacements

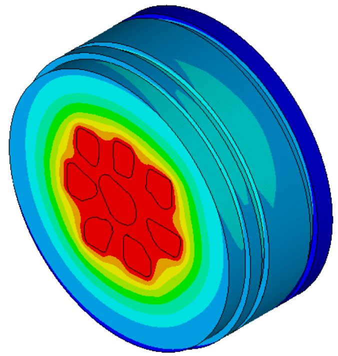


Residual stress distribution



FE model of the die

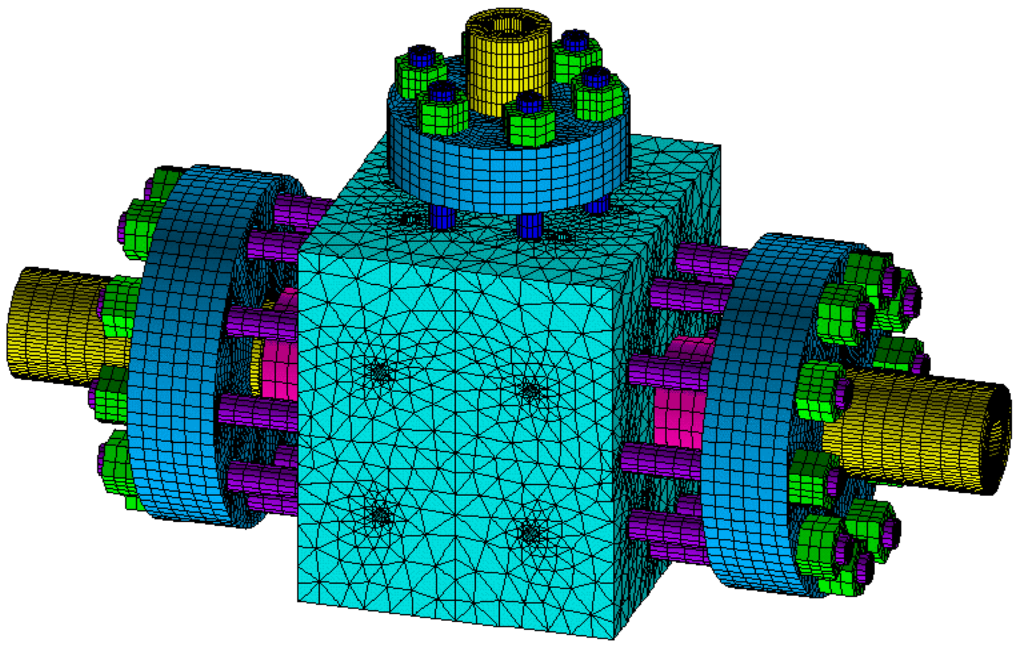




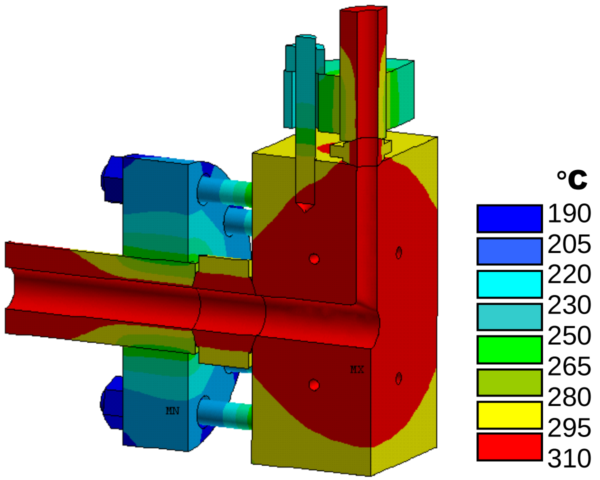
Velocity and temperature distribution

FE analysis of a high pressure T-connection

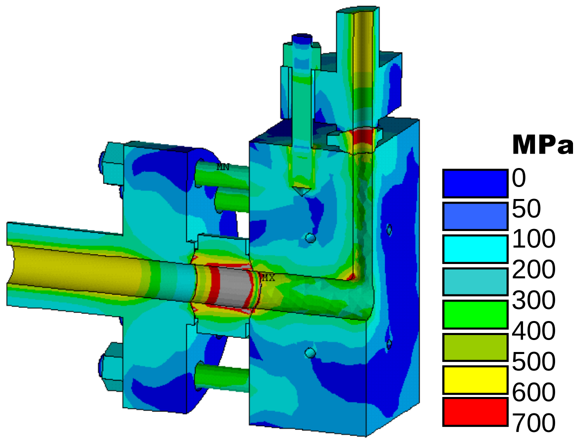
The aim of the analysis was to find out stress and strain distribution in a T-connection caused by high internal pressure (2600 at) and temperature gradients. External cooling, assembly procedure (screw pretension), contact and plasticity effects have been included. The project done for ORLEN petrochemical company



FE model



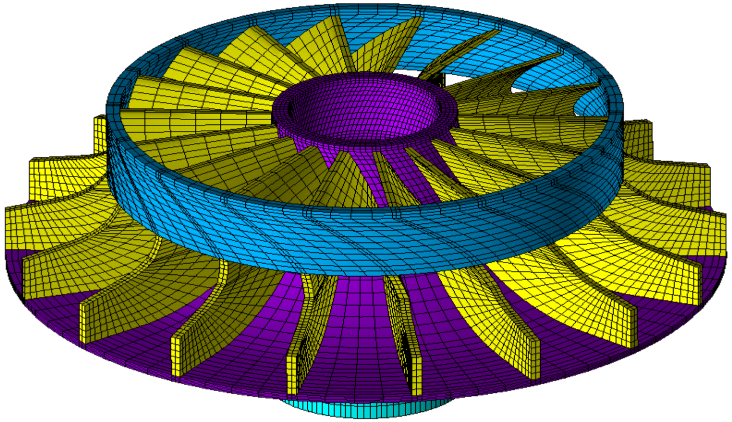
Temperature distribution



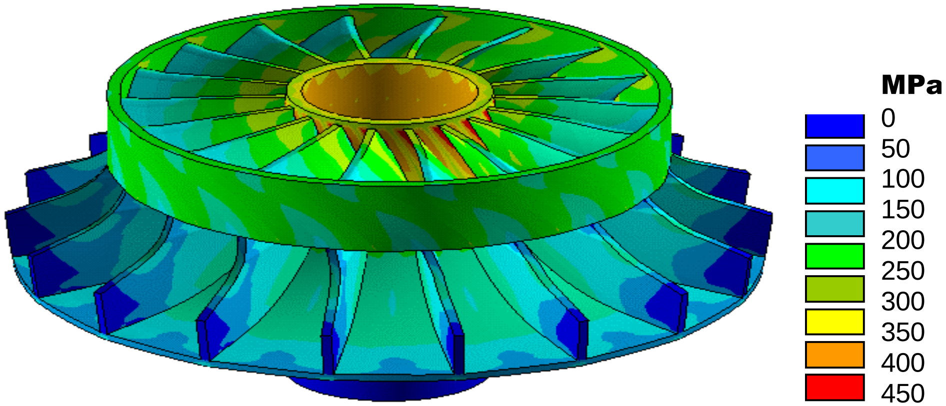
Von Mises stress

FE analyses of rotor disks

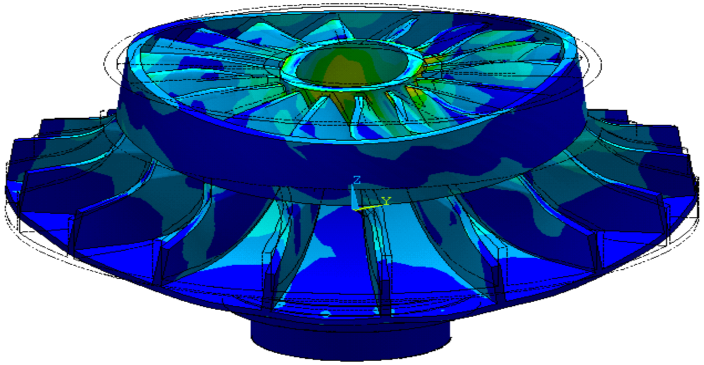
The aim of the analysis was to asses the right shape details of the rotor to avoid high stresses and to find its vibration characteristics.



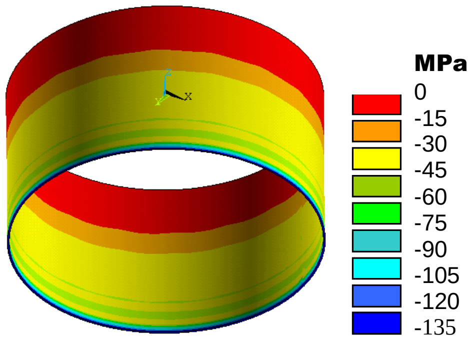
FE mesh



Von Mises stress distribution



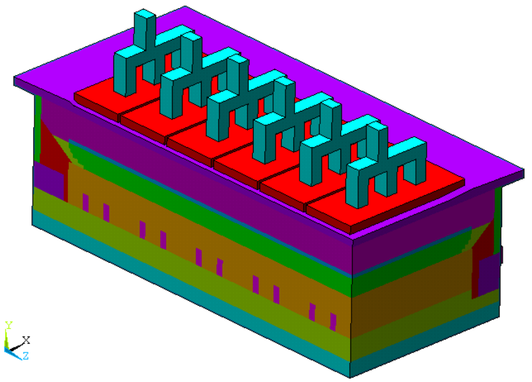
The mode shape for the natural frequency of 2203Hz



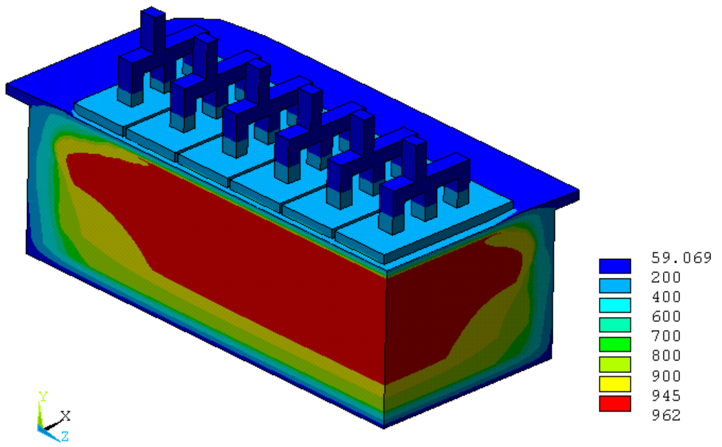
Contact pressure between the shaft and the rotor disk

Thermo-electrical analysis of aluminium reduction cells

The analyses were performed to find temperature field and electrical potential distribution inside the reductant cell used in the process of aluminium production. The project done for Alusuisse Technological Center, Sierre, Switzerland.

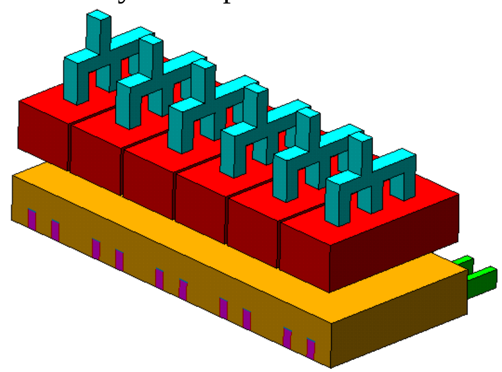


FE model (*quarter of the cell*)

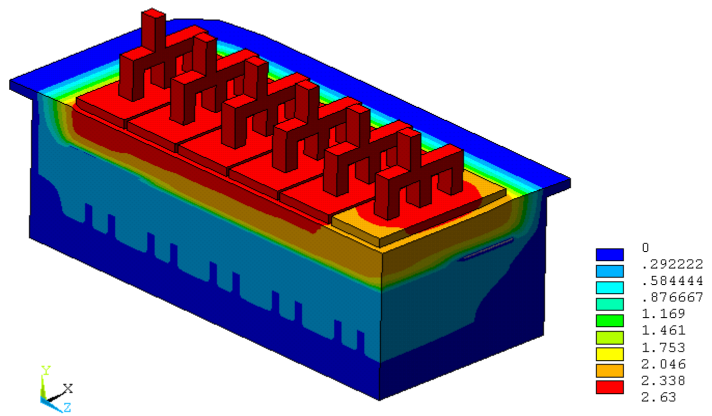


Temperature distribution

The influence of geometry, material properties and boundary conditions on the phenomena that take place in the bath and liquid aluminium is investigated. The analysis enabled to correct the design and to improve efficiency of the processes.



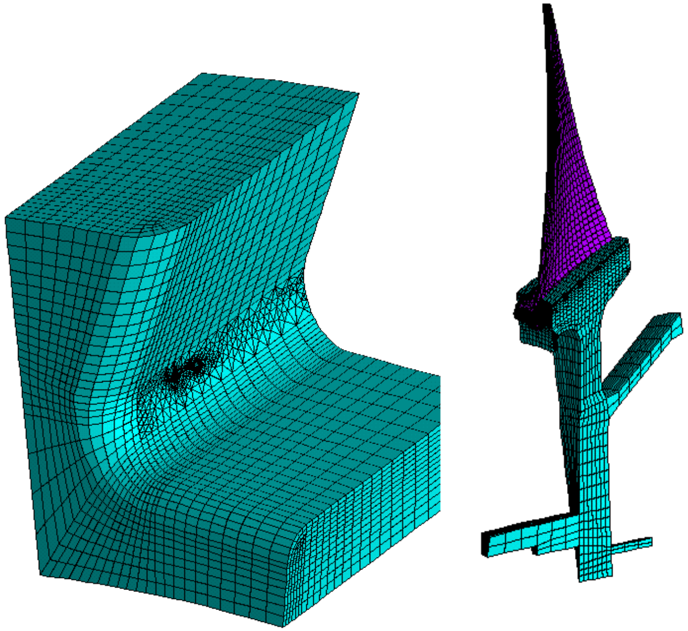
Anode and cathode blocks



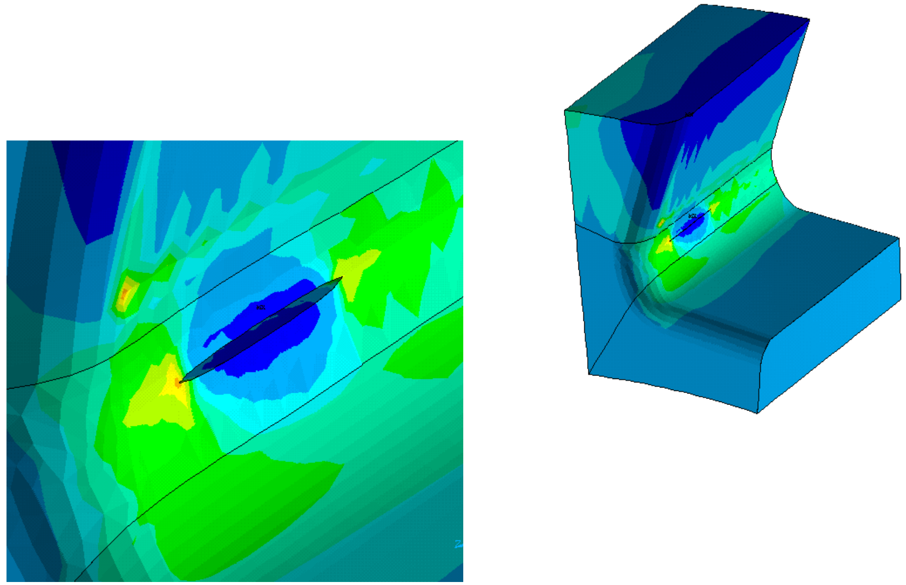
Electric potential distribution

FE analysis of the turbine blade locking part defects (imperfections)

Experiments show the presence of defects like surface scratch, or micro-crack in the region of blade locking part of the turbine disks. Such imperfections may result in crack initiation and propagation. A segment of the turbine disk together with a blade has been modelled (including contact). Half-elliptical crack has been introduced in the sub-model. Stress intensity factors and Rice integral values have been calculated.



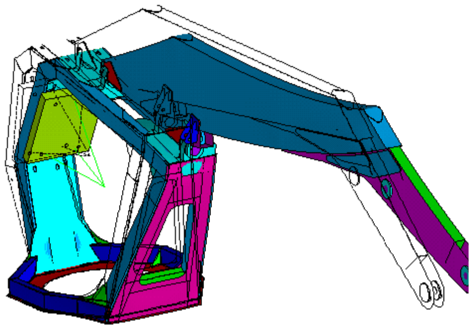
FE model

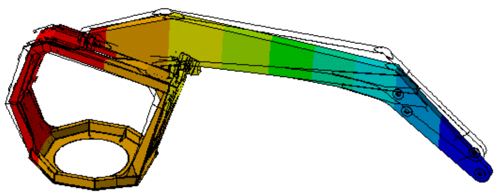


Von Mises stress distribution in the vicinity of the crack

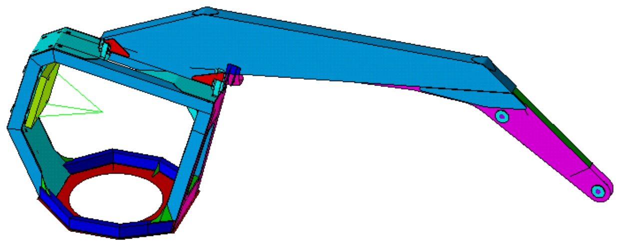
FEM Analysis of the Winch Frame and Boom of the snow groomer

The aim of the analysis was to check the stiffness and stress level of the new design of the structure. Numerical model consisted of FE shell elements supplemented by brick, beam, link and mass elements. In regions of special care sub-models were used involving contact elements. The results suggested essential changes of design. The project done for PLUMETTAZ S.A., Bex, Switzerland

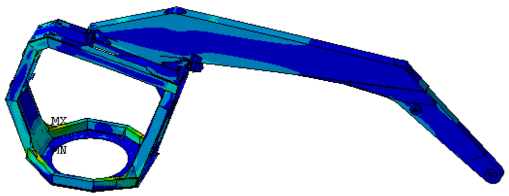




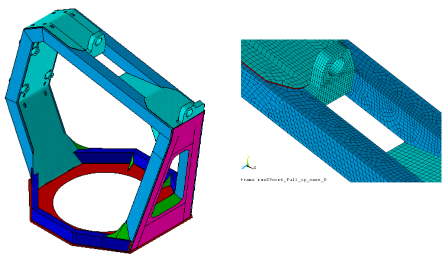
Displacements



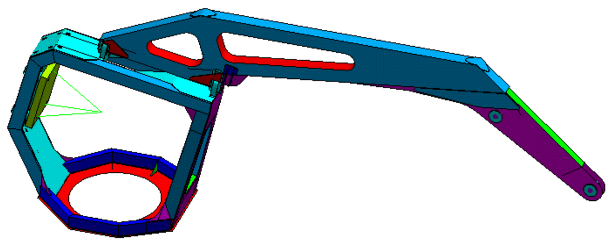
Initial model



Von Mises stress

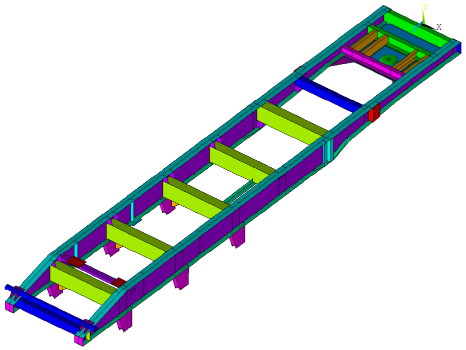


FE model

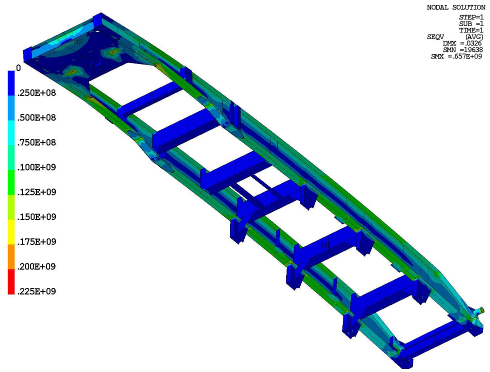


Modified design

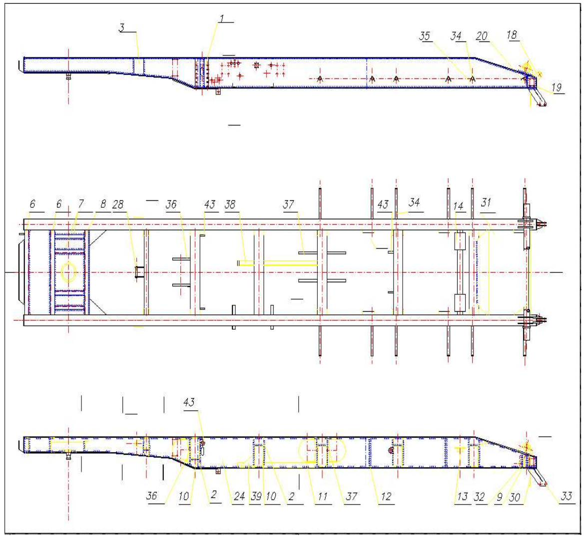
CAD/CAE study of a New Design of Truck Frame

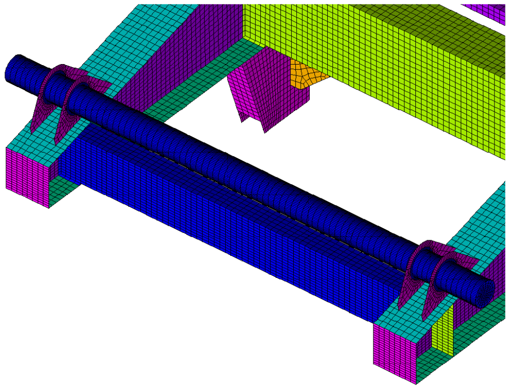


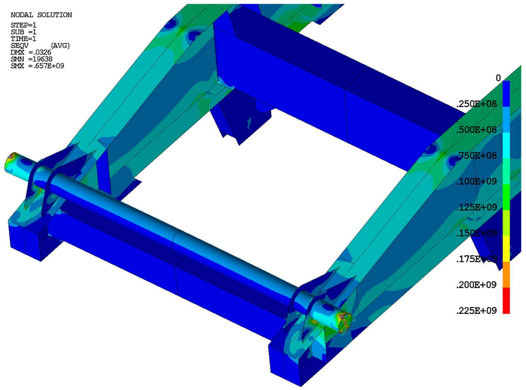
FE model



Von Mises stress



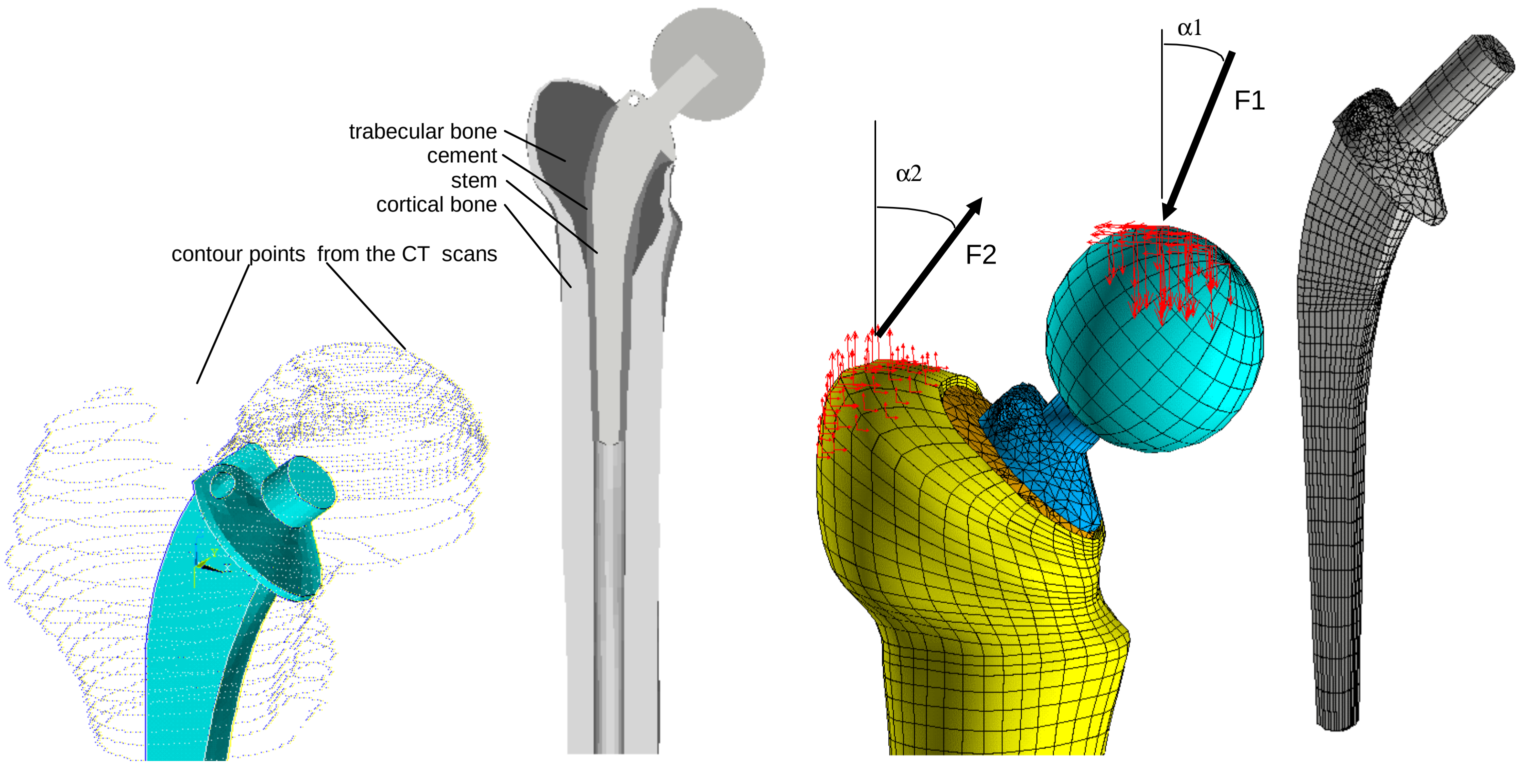




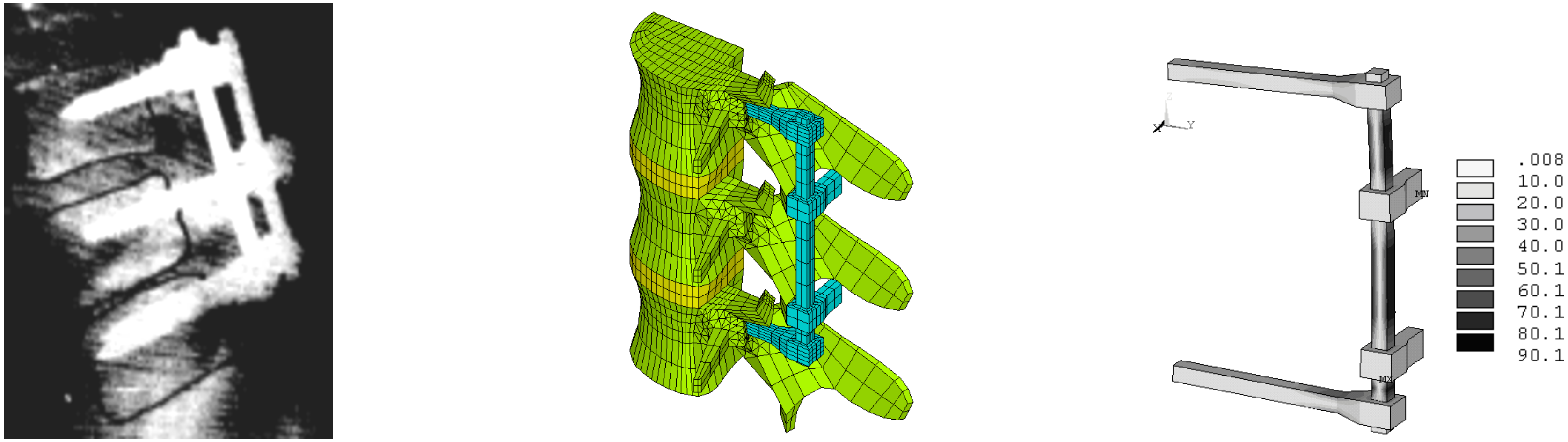
CAD project

Finite element method in bone-implant system strength analysis

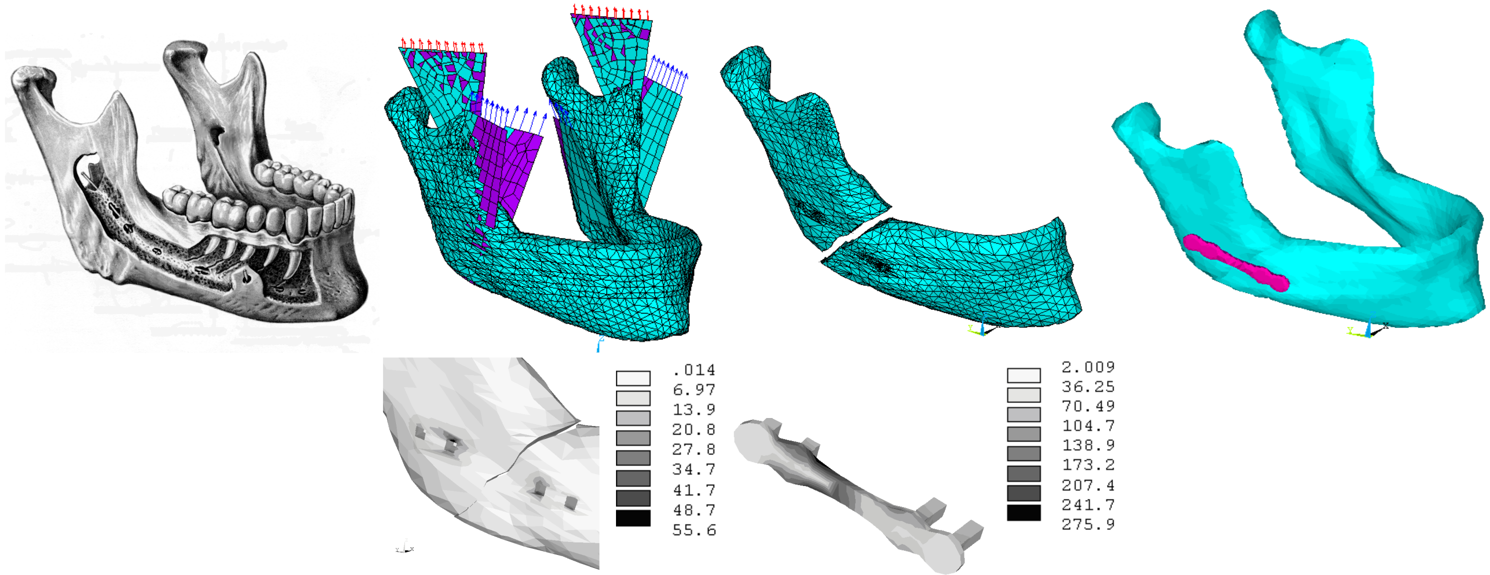
The three-dimensional FE models of the living tissues-implant systems can deliver the valuable information about mechanisms of stress transfer and healing processes after the orthopaedic surgery. In the presented example some different variants of the hip stem were considered to find the best solution, which should reduce stress concentration within the bone tissues. The model of the femur was built using the data obtained from CT scans. The considered load corresponds to one leg stance of a man weighting 800N.



Finite element model of the femur endoprosthesis : *body weight BW=800N, F1=2.47 BW, F2=1.55BW,* $\alpha 1=28°$*,* $\alpha 2= 40°$



FE model of the spine stabilizer and the von Mises stress distribution within the frame



Von Mises Equivalent stress (MPa)

FE model and selected results of numerical simulation of mandibular ostheosynthesis

3. Finite Difference Method (FDM) Boundary Element Method (BEM) and Finite Element Method (FEM)

Draft presentation for solving Poisson’s equation in 2D space

Poisson's equation is a partial differential equation with broad utility in electrostatics, mechanical engineering and theoretical physics.

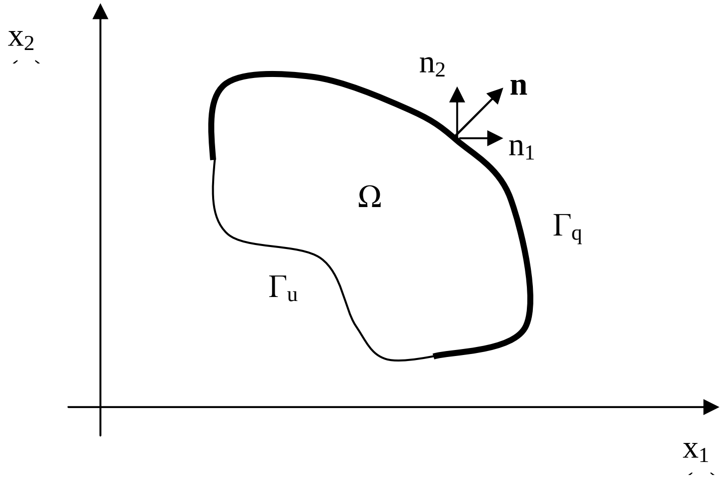
$$\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+f(x_1,x_2)=0$$

For vanishing f, this equation becomes Laplace's equation.

We consider a Dirichlet boundary condition on Γ_u and a Neumann boundary condition on Γ_q:

$$u(\overline{x})=u_0 \quad , \quad \overline{x}\in\Gamma_u$$

$$q(x)=\frac{\partial u(\overline{x})}{\partial n}=q_0 \quad , \quad \overline{x}\in\Gamma_q$$



where u_0 and q_0 are given functions defined on those portions of the boundary.
In some simple cases (shape of the domain Ω and boundary conditions) the Poisson equation may be solved using analytical methods.

Finite Difference Method

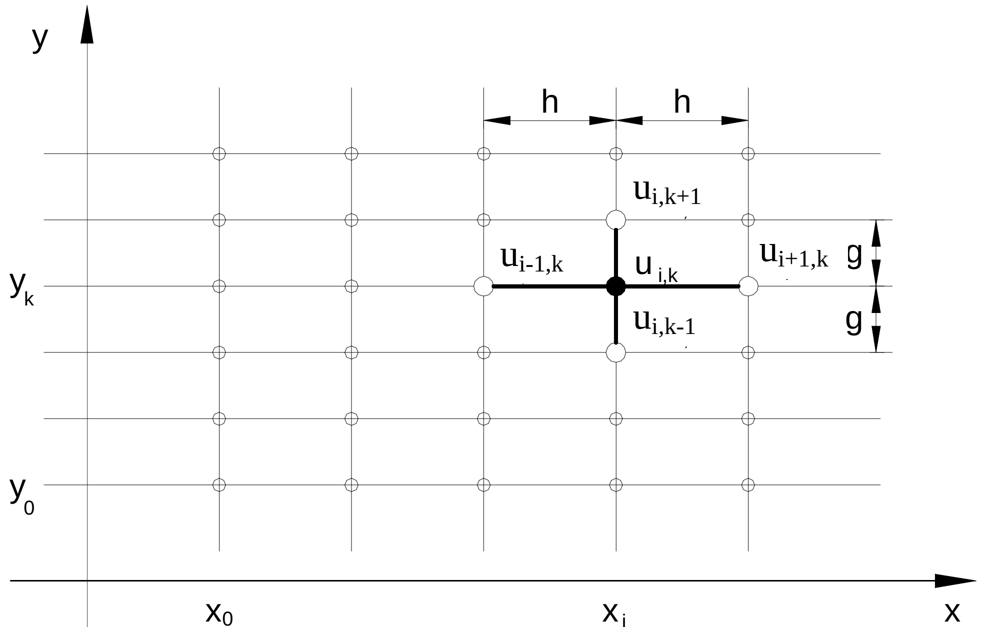
Finite-difference method approximates the solution of differential equation by replacing derivative expressions with approximately equivalent difference quotients. That is, because the first derivative of a function f (x) is, by definition,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

then a reasonable approximation for that derivative would be to take

$$f'(a) \approx \frac{f(a+h)-f(a)}{h} \quad \text{(difference quotient)}$$

for some small value of h. Depending on the application, the spacing *h* may be variable or held constant.
The approximation of derivatives by finite differences plays a central role in finite difference methods
In similar way it is possible to approximate the first **partial derivatives** using **forward , backward** or central **differences**



$$u_{i,k} = u(x_i, y_k)$$

a) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k}}{g},$$

b) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k} - u_{i,k-1}}{g},$$

c) $$\frac{\partial u}{\partial y} \approx \frac{\Delta u}{\Delta y} = \frac{u_{i,k+1} - u_{i,k-1}}{2g}.$$

Differences corresponding to higher derivatives

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{u_{i+1,k} - 2u_{i,k} + u_{i-1,k}}{h^2},$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{\Delta^2 u}{\Delta y^2} = \frac{u_{i,k+1} - 2u_{i,k} + u_{i,k-1}}{g^2}.$$

$$\frac{\partial^4 u}{\partial x^4} \approx \frac{\Delta^4 u}{\Delta x^4} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4}$$

Using the finite differences we can approximate the partial differential equation at any point (x_i, y_j) by an algebraic equation .
In the case of Poissons equation:

$$\frac{1}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + \frac{1}{g^2}\left(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}\right) + f\left(x_i, y_j\right) = 0.$$

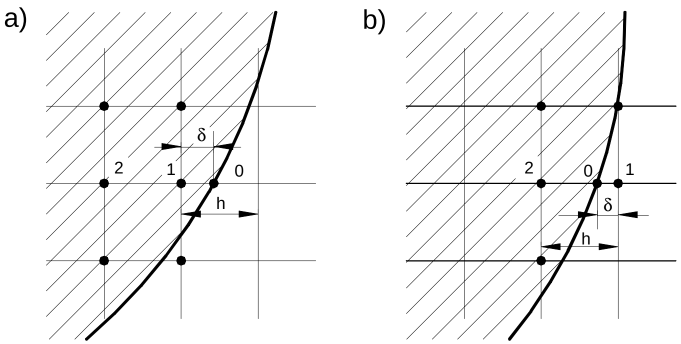
If $h = g$ i $f \equiv 0$ (Laplace equation) we get

$$u_{i,j} = \frac{u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}}{4}.$$

N grid points in the domain Ω , N equations, N unknows

$$[A]\{u\} = \{b\}$$

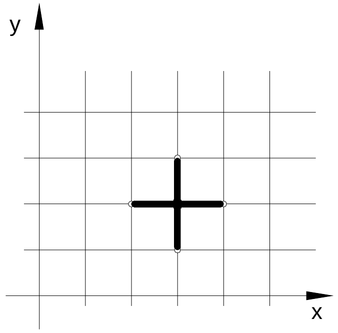
discrete form of boundary conditions

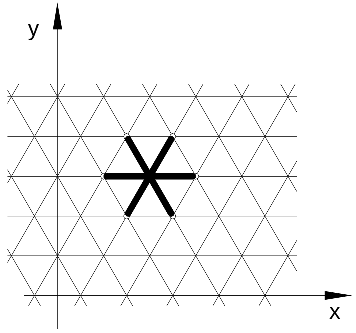


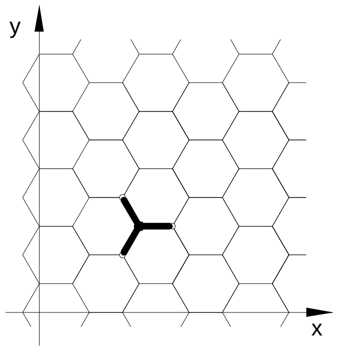
In the case of irregular boundary shape

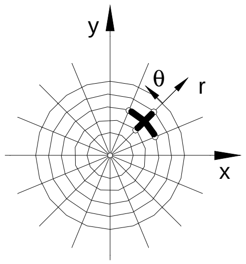
a) assumed $u_1 = \dfrac{hu_0 + \delta u_2}{h + \delta}$ instead of $u = u_0$

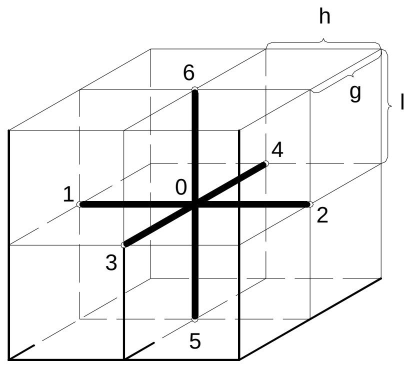
b) assumed $u_1 = \dfrac{hu_0 - \delta u_2}{h - \delta}$ instead of $u = u_0$


y
x


y
x


y
x


y
θ
r
x

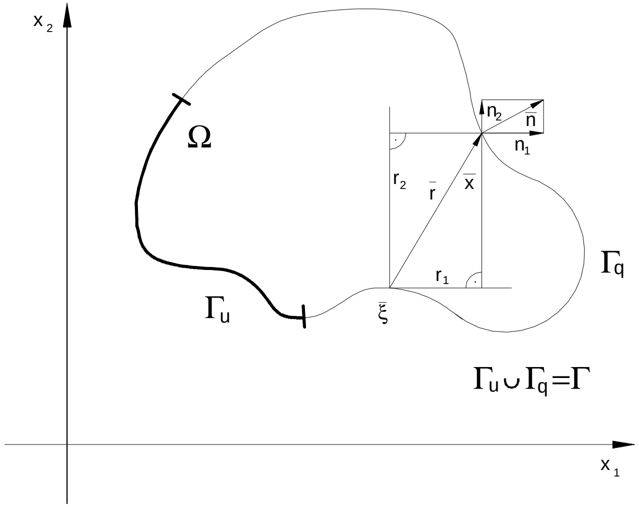

h
6
g
l
4
0
1
2
3
5

Boundary Element Method

Uses the boundary integral equation (equivalent to the Poisson’s equation with the adequate b.c.)

$$c(\overline{\xi})u(\overline{\xi}) = -\int_{\Gamma} u(x)q^{*}(\overline{\xi},\overline{x})d\Gamma(x) + \int_{\Gamma} \frac{\partial u(\overline{x})}{\partial \overline{n}} u^{*}(\overline{\xi},\overline{x})d\Gamma(\overline{x}) + \int_{\Omega} f(x)u^{*}(\overline{\xi},\overline{x})dR(\overline{x})$$

$c(\overline{\xi})$ - coefficient equal to 1/2 on the smooth contour, 1 inside the domain Ω



Kernel functions $$u^{*} = (\overline{\xi},\overline{x}) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right),$$

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2}\ .$$

$$q^{*}(\overline{\xi},\overline{x}) = \frac{\partial u^{*}(\overline{\xi},\overline{x})}{\partial n}\ .$$

$$q^{*} = \frac{\partial u^{*}}{\partial x_1}\cdot n_1 + \frac{\partial u^{*}}{\partial x_2}\cdot n_2,$$

$$q^{*} = \frac{-(r_1\cdot n_1 + r_2\cdot n_2)}{2\pi r^2},$$

$$\frac{\partial r}{\partial x_i} = \frac{x_i - \xi_i}{r} = \frac{r_i}{r}\ .$$

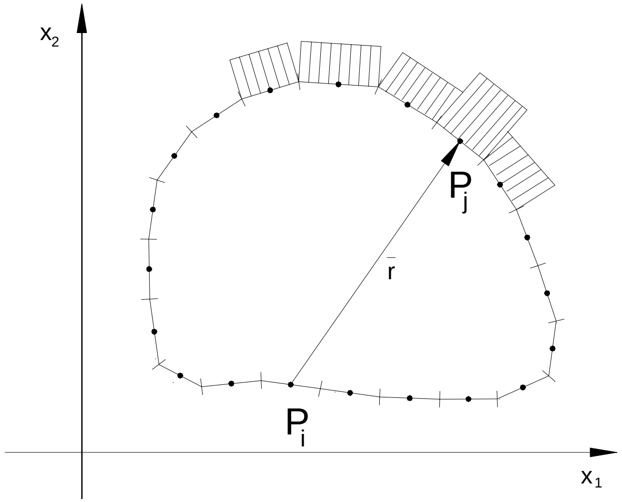
The boundary integral equation states the relation between $u(\overline{x})$ and its derivative in normal direction $q(\overline{x}) = \frac{\partial u(\overline{x})}{\partial \overline{n}}$ on the boundary Γ.

The numerical approach

1. **Discretization of the boundary (LE boundary elements)**
2. **Approximation of** $u(\overline{x})$ and $q(\overline{x})$ on the boundary

(e.g. u(P_i), q(P_i) constant on boundary elements)

3 . **Building the set of linear equations**



$$\frac{1}{2}u(P_i) = \sum_{j=1}^{LE}\int_{\Gamma_j} u^*(P_i,\overline{x})q(P_j)d\Gamma_j - \sum_{j=1}^{LE}\int_{\Gamma_j} q^*(P_i,\overline{x})u(P_j)d\Gamma_j$$

$$+\int_{\Omega} f(\overline{x})u^*(P_i,\overline{x})dR \qquad i=1,2,..LE$$

$$\frac{1}{2}u(P_i) = \sum_{j=1}^{LE}U_{ij}^*\cdot q(P_j) - \sum_{j=1}^{LE}Q_{ij}^*\cdot u(P_j) + f_i, \qquad i=1,2...LE\,.\; f_i = \int_{\Omega} f(\overline{x})u^*(P_i,\overline{x})d\Omega(\overline{x})$$

$$\frac{1}{2}\{u\} = \left[U^*\right]\{q\} - \left[Q^*\right]\{u\} + \{f\}.$$

LE linear equations with the unknows $u(P_j)$ (if the point $P_j \in \Gamma_q$) or $q(P_i)$ (if $P_i \in \Gamma_u$)

Finally: $[A]\{y\}=\{b\}$

The solution {y} represents unknown boundary values of u and q.

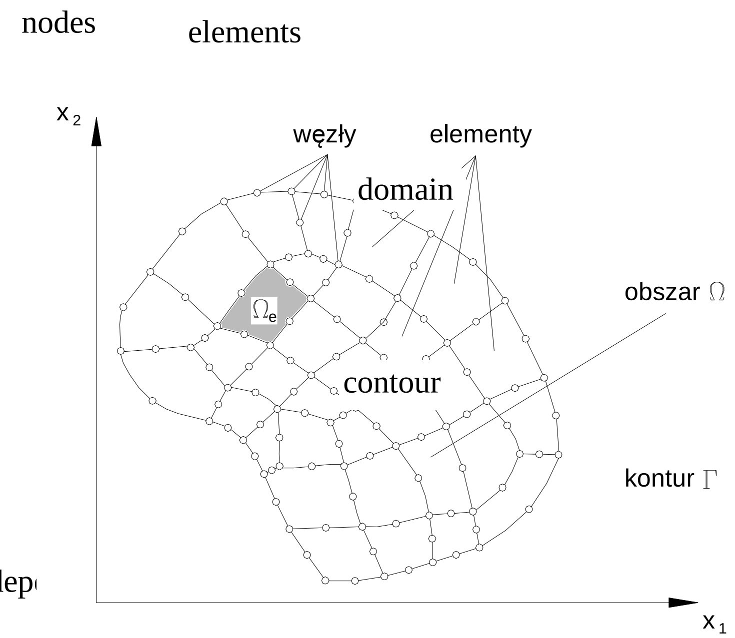
The matrix A – full, unsymmetric

4. **Solution** - provides complete information about the function $u(\overline{x})$ and its derivative $q(\overline{x})$ on the boundary

Boundary Element Method reduces the number of unknown parameters (DOF of the discrete model) in comparison to FDM and FEM (domain methods).

Finite Element Method

Equivalent problem of minimising of the fuctional:



$$I(u)=\frac{1}{2}\int_{\Omega}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega-\int_{\Gamma_q}q_0 u\, d\Gamma,$$

with the Dirchlet b. c.

$$u(\overline{x})=u_0, \qquad \overline{x}\in\Gamma_u$$

1. Discretization of the solution domain Ω into elements Ω_i, i=1,LE,

connected in the nodes

$$\Omega=\bigcup_{i=1}^{LE}\Omega_e \quad \text{i} \quad \Omega_i\cap\Omega_j=0 \qquad i\neq j,$$

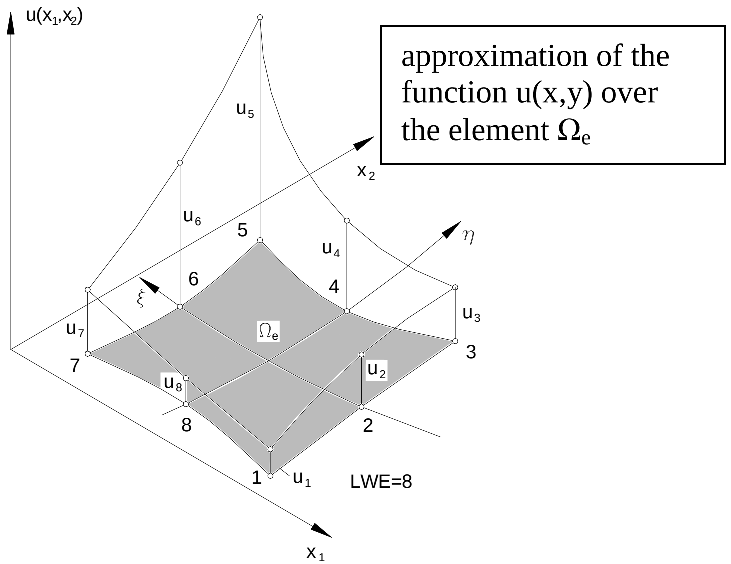
2.Approximation of function $u(\overline{x})$ within the finite element in the form of polynomials dep

$$u(x_1,x_2)=\sum_{i=1}^{LWE}N_i(x_1,x_2)u_i$$

LWE – number of nodes of the element

u_i, i = 1,...,LWE - nodal values of the approximated function,

$N_i(x_1,x_2)$ – shape functions



3. Discrete form of the functional

$$I(u)\cong\sum_{i=1}^{LE}\frac{1}{2}\int_{\Omega_i}\left[\left(\frac{\partial u}{\partial x_1}\right)^2+\left(\frac{\partial u}{\partial x_2}\right)^2-2f(x_1,x_2)u\right]d\Omega_i-\sum_{j=1}^{LK}\int_{\Gamma_j}q_0 u\, d\Gamma_j$$

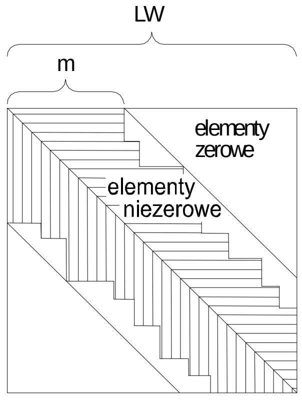
In each element

$$\frac{\partial u}{\partial x_1} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_1} u_i ,$$

$$\frac{\partial u}{\partial x_2} = \sum_{i=1}^{LWE} \frac{\partial N_i}{\partial x_2} u_i .$$

In this way the functional **I** is replaced by the function of several unknows u_i, $i = 1, 2, ..., LW$, where LW denotes the number of nodes of the finite element mesh. In the matrix form :

$$I(u) \approx \frac{1}{2} \lfloor u_1, u_2, u_3, ..., u_{LW} \rfloor \begin{bmatrix} k_{11} & k_{12} & k_{13} & \dots & k_{1LW} \\ k_{21} & k_{22} & k_{23} & & \\ k_{31} & k_{32} & & & \\ \dots & & & & \\ k_{LW1} & & & & k_{LW\,LW} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ \dots \\ u_{LW} \end{Bmatrix} - \lfloor u_1, u_2, u_3, \dots, u_{LW} \rfloor \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \\ \dots \\ b_{LW} \end{Bmatrix}$$



$$I \approx \frac{1}{2} \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times LW}{[K]} \underset{LW\times 1}{\{u\}} - \underset{1\times LW}{\lfloor u \rfloor} \underset{LW\times 1}{\{b\}} \cdot$$

Necessary (and sufficient) condition of the minimum:

$$\frac{\partial I}{\partial u_i} = 0, \qquad i = 1, \dots, LW .$$

matrix: sparse, symmetrical, positive defined, banded

Hence

$$[K]\{u\} = \{b\}, \text{ (+ Dirichlet b.c.)}$$

Set of the simultaneuous equations with unknown nodal values of the investigated function.

4. BEAMS
RITZ-RAYLAYGH METHOD and FINITE ELEMENT METHOD

Principle of minimum potential energy.

The **potential energy** of an elastic body is defined as

Total potential energy (V)= Strain energy (U) – potential energy of loading (Wz)

In theory of elasticity the potential energy is the sum of the elastic energy and the work potential:

$$V = U - W_z = \frac{1}{2}\int_{\Omega} \sigma_{ij}\varepsilon_{ij}\, d\Omega - \int_{\Omega} X_i u_i\, d\Omega - \int_{\Gamma} p_i u_i\, d\Gamma$$

Ω – domain of the elastic body, Γ – boundary, σ_{ij} – stress state tensor, ε_{ij} – strain state tensor ,

u_i – displacement vector, p_i – boundary load (traction), X_i– body loads

The potential energy is a functional of the displacement field. The body force is prescribed over the volume of the body, and the traction is prescribed on the surface Γ. The first two integral extends over the volume of the body. The third integral extends over the boundary.

The principle of minimum potential energy states that,
the displacement field that represents the solution of the problem fullfills the displacement boundary conditions and inimizes the total potential energy.

$$V = U - W_z = \min!,$$

Total potential energy of the beam loaded by the distributed load $p\left[\frac{\text{N}}{\text{m}}\right]$:

$$V = \frac{1}{2}\int_0^l EI(w'')^2\,dx - \int_0^l pw\,dx\,,$$

where the function w(x) describes deflection of the beam

Ritz method

1.The problem must be stated in a variational form,
as a minimization problem, that is:
find w(x) minimizing the functional V(w)

2. The solution is approximated by a finite linear combination of the form:

$$\tilde{w}(x) = \sum_{i=1}^{n} a_i \varphi_i(x)$$

where a_i denote the undetermined parameters termed **Ritz coefficients**,
and φi are the assumed **approximation functions** (i=1,2...n).
The approximate functions φi must be linearly independent
and

3. Finally functional V is approximated by the function of n variables a_i

$V=V(a_1, a_2, a_3, \ldots a_n)$

3. The parameters a_i are determined by requirement that the functional is minimized with respect to a_i

$$\frac{\partial V}{\partial a_i} = 0\,, \qquad i = 1, \ldots, n\,.$$

$$[A]\{a\} = \{b\}$$

4. The solution provides a_i, and the approximate solution

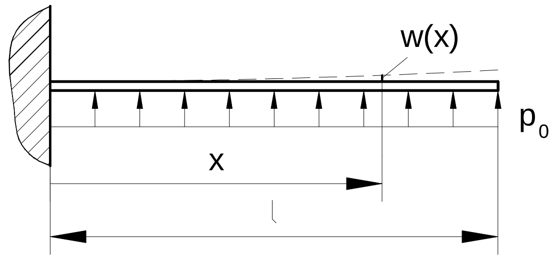
$$\tilde{w}(x) = \sum_{i=1}^{n} a_i \varphi_i(x)\;.$$

Hence the approximate internal forces in the beam

$$\tilde{M}_q(x) = EI\tilde{w}''(x),$$
$$\tilde{T}(x) = -EI\tilde{w}'''(x).$$

EXAMPLE

Find the deflection of the cantilever beam under the load $p_0\left[\frac{N}{m}\right]$ using the analytical solution of the differential equation and compare it to the approximate solution using Ritz method with the function $\tilde{w}(x)=a_1+a_2x+a_3x^2+a_4x^3$.



Exact analytical solution

$$w''(x)=\frac{M_g(x)}{EI} \qquad M_q(x)=\frac{p_0}{2}(l-x)^2,$$

$$w(x)=0, \quad d\frac{w(x)}{dx}=0$$

Solution

$$w(x)=\frac{p_0}{24EI}(6l^2-4lx+x^2)x^2,$$

Max. deflection w(l)= $p_0\, l^2$ /8 EI

The approximate solution $\tilde{w}(x)=a_1+a_2x+a_3x^2+a_4x^3$ has to satisfy the displacement boundary conditions

$$\tilde{w}(x=0)=0, \qquad \tilde{w}'(x=0)=0.$$

Thus

$$\tilde{w}(x)=a_3x^2+a_4x^3.$$

$$V=\frac{EI}{2}(4a_3^2l+12a_3a_4l^2+12a_4^2l^3)-p(a_3\frac{l^3}{3}+a_4\frac{l^4}{4}).$$

$$\frac{\partial V}{\partial a_3} = \frac{EI}{2}\left(8la_3 + 12l^2 a_4\right) - \frac{p_0 l^3}{3} = 0,$$

$$\frac{\partial V}{\partial a_4} = \frac{EI}{2}\left(12l^2 a_3 + 24l^3 a_4\right) - \frac{p_0 l^4}{4} = 0.$$

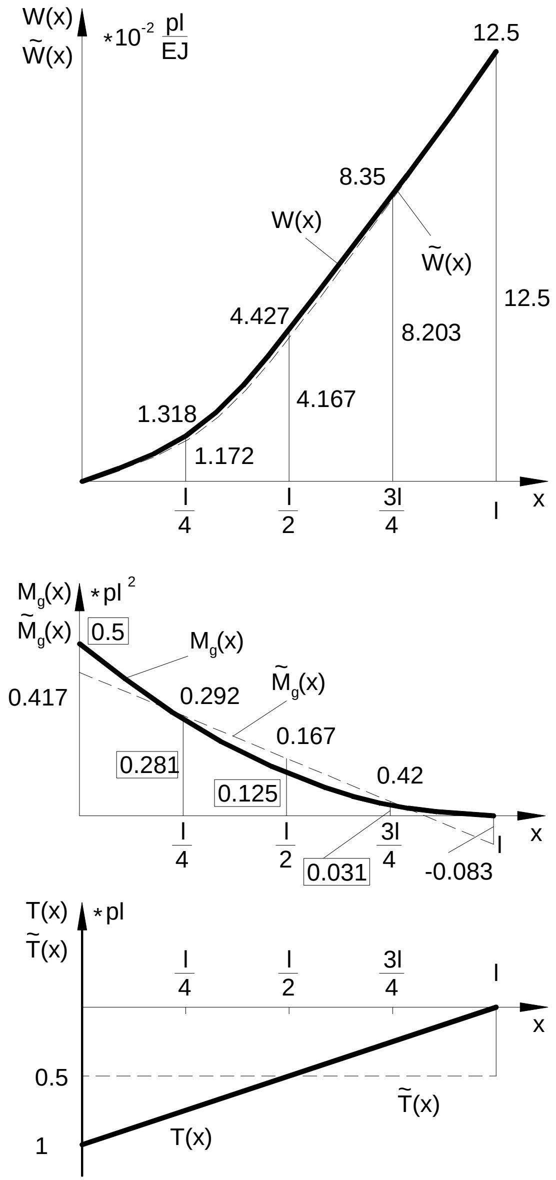
$$a_3 = \frac{5}{24}\frac{p_0 l^2}{EI}, \quad a_4 = -\frac{p_0 l}{12EI}$$

Finally the approximate solution is

$$\tilde{w}(x) = \frac{5}{24}\frac{p_0 l^2}{EI}x^2 - \frac{p_0}{12EI}x^3,$$

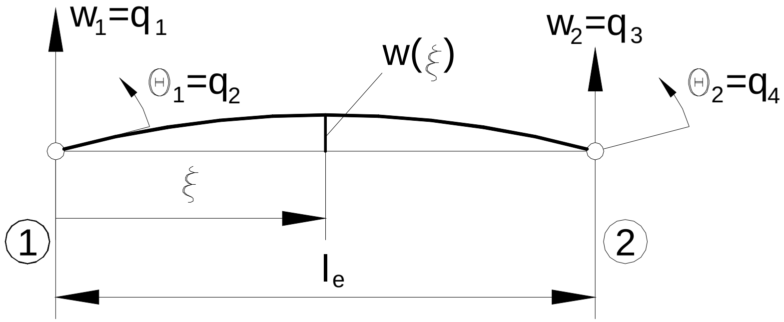
$$\tilde{M}_q(x) = \frac{5}{12}p_0 l^2 - \frac{p_0 l}{2}x,$$

$$\tilde{T}(x) = \frac{-p_0 l}{2}.$$



Graphs presenting exact (bold line) and approximate (dashed line) solutions
of the cantilever beam:
displacement, bending moment, shear force

Approximation : local, with nodal displacements w_1 , w_2 , θ_1 *and* θ_2 as unknown parameters



Positive directions:
upward for translation
counter clockwise for rotation

Simple beam finite element

Lets assume first the polynomial approximation within the element

$$w(\xi) = \alpha_1 + \alpha_2 \xi + \alpha_3 \xi^2 + \alpha_4 \xi^3$$

with four unknown parameters α_i .

The required new parameters : nodal displacements w_1 , w_2 , θ_1 *and* θ_2 (degrees of freedom – DOF of the element)

Nodal displacement vector

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix} .$$

$$w(\xi) = \sum_{i=1}^{4} N_i(\xi) q_i \ .$$

$$w(\xi) = \lfloor N(\xi) \rfloor \{q\}_e \ ,$$

<u>Relation between $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and q_1,q_2,q_3,q_4</u>

$$q_1 = w(0) = \alpha_1,$$ displacement and node 1

$$q_2 = \frac{dw}{d\xi}(0) = \alpha_2,$$ slope at node 1

$$q_3 = w(l) = \alpha_1 + \alpha_2 l_e + \alpha_3 l_e^2 + \alpha_4 l_e^3,$$ displacement at node 2

$$q_4 = \frac{dw}{d\xi}(l) = \alpha_2 + 2\alpha_3 l_e + 3\alpha_4 l_e^2.$$ slope at node 2

In the matrix form

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l_e & l_e^2 & l_e^3 \\ 0 & 1 & 2l_e & 3l_e^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix}.$$

$$\begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \begin{bmatrix} & & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-3}{l_e^2} & \frac{-2}{l} & \frac{3}{l_e^2} & \frac{-1}{l_e} \\ \frac{2}{l_e^3} & \frac{1}{l_e} & \frac{-2}{l_e^3} & \frac{1}{l_e^2} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e$$

The approximate deflection may be presented in the form

$$w(\xi) = \lfloor 1, \xi, \xi^2, \xi^3 \rfloor \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{Bmatrix} = \lfloor N_1(\xi), N_2(\xi), N_3(\xi), N_4(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix},$$

$$N_1(\xi) = 1 - 3\frac{\xi^2}{l_e^2} + 2\frac{\xi^3}{l_e^3},$$

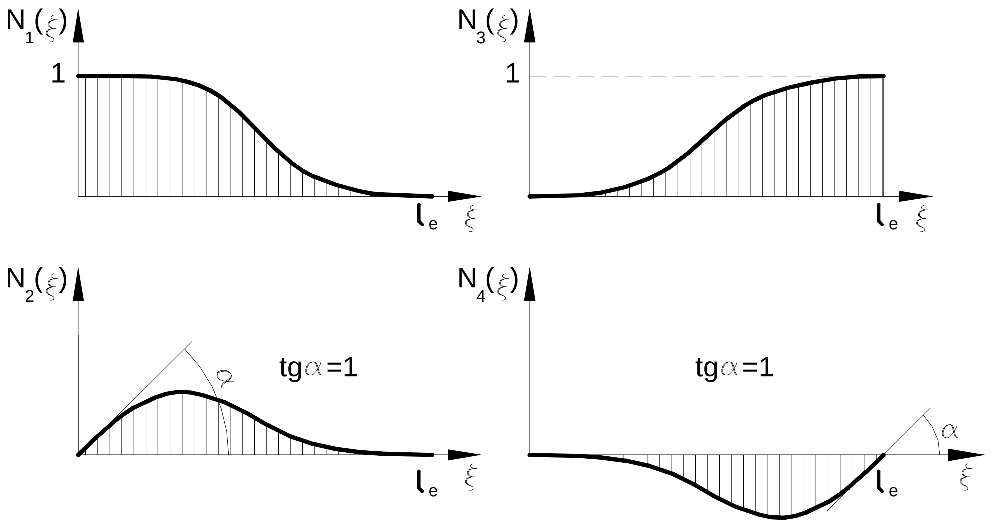
$$N_2(\xi) = \xi - 2\frac{\xi^2}{l_e} + \frac{\xi^3}{l_e^2},$$

$$N_3(\xi) = 3\frac{\xi^2}{l_e^2} - 2\frac{\xi^3}{l_e^3},$$

$$N_4(\xi) = \frac{-\xi^2}{l_e} + \frac{\xi^3}{l_e^2}.$$

The functions $N_i(\xi)$ are called **shape functions** of the beam element.

$N_i(\xi)$ describes deflection of the beam element , where $q_i = 1$, and for $j \neq i \quad q_j = 0$ (see graphs).



Shape functions of a beam element

$$w(\xi) = \lfloor N(\xi) \rfloor \{q\}_e \,,$$
$$w'(\xi) = \lfloor N'(\xi) \rfloor \{q\}_e \,,$$
$$w''(\xi) = \lfloor N''(\xi) \rfloor \{q\}_e \,.$$

Total potential energy of the beam element of the length l_e

$$V_e = U_e - W_{ze} = \frac{EI}{2} \int_0^{l_e} (w''(\xi))^2 \, d\xi - \int_0^{l_e} p(\xi) w(\xi) d\xi \,.$$

$$U_e = \frac{EI}{2}\int_0^{l_e} w''(\xi)w''(\xi)d\xi = \frac{EI}{2}\int_0^{l_e} \lfloor q \rfloor_e \{N''\}\lfloor N'' \rfloor \{q\}_e \, d\xi =$$

$$= \frac{EI}{2}\lfloor q \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} d\xi \{q\}_e .$$

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e , \qquad [k]_e = EI \begin{bmatrix} \int_0^{l_e} N_1''N_1''d\xi & \int_0^{l_e} N_1''N_2''d\xi & \int_0^{l_e} N_1''N_3''d\xi & \int_0^{l_e} N_1''N_4''d\xi \\ \int_0^{l_e} N_2''N_1''d\xi & \int_0^{l_e} N_2''N_2''d\xi & \int_0^{l_e} N_2''N_3''d\xi & \int_0^{l_e} N_2''N_4''d\xi \\ \int_0^{l_e} N_3''N_1''d\xi & \int_0^{l_e} N_3''N_2''d\xi & \int_0^{l_e} N_3''N_3''d\xi & \int_0^{l_e} N_3''N_4''d\xi \\ \int_0^{l_e} N_4''N_1''d\xi & \int_0^{l_e} N_4''N_2''d\xi & \int_0^{l_e} N_4''N_3''d\xi & \int_0^{l_e} N_4''N_4''d\xi \end{bmatrix}$$

Matrix $[k]_e$ is named stiffness matrix of beam element. After integration

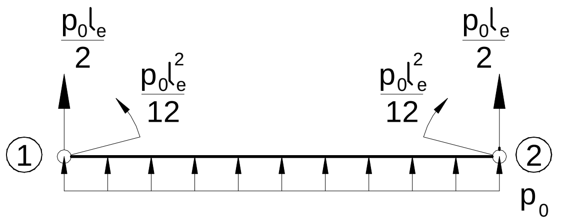
$$[k]_e = \frac{2EI}{l_e^3} \begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix} .$$

The external work done by the traction p:

$$W_{ze}^{p} = \int_{0}^{l_e} p(\xi) w(\xi) d\xi = \int_{0}^{l_e} p(\xi) \lfloor N(\xi) \rfloor \{q\}_e \, d\xi = \int_{0}^{l_e} \lfloor N_1(\xi) p(\xi) d\xi, N_2(\xi) p(\xi) d\xi, N_3(\xi) p(\xi) d\xi, N_4(\xi) p(\xi) d\xi \rfloor \{q\}_e \, d\xi,$$

$$W_{ze}^{p} = \lfloor F_1^e, F_2^e, F_3^e, F_4^e \rfloor_e \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix} = \lfloor F \rfloor_e \{q\}_e , \qquad F_i^e = \int_{0}^{l_e} N_i(\xi) p(\xi) d\xi$$

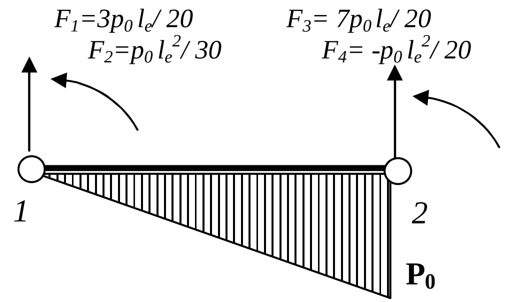
F_i^e - equivalent nodal forces



$$F_1^e = F_3^e = \frac{p_0 l_e}{2}$$

$$F_2^e = \frac{p_0 l_e^2}{12}$$

$$F_4^e = \frac{-p_0 l_e^2}{12}$$



Equivalent nodal forces corresponding to the constant and linear distribution of p_0 *load*
(kinematically equivalent or work-equivalent !)

Total potential energy of the beam element

$$V_e = U_e - W_{ze} = \frac{1}{2} \underset{1\times4}{\lfloor q \rfloor_e} \underset{4\times4}{[k]_e} \underset{4\times1}{\{q\}_e} - \underset{1\times4}{\lfloor q \rfloor_e} \underset{4\times1}{\{F\}_e} .$$

The conditions for finding the minimum of V_e:

$$\frac{\partial V_e}{\partial q_i} = 0, \qquad i = 1, 2, 3, \ldots, n$$

$$[k]_e \{q\}_e = \{F\}_e .$$

$$\frac{2EI}{l_e^3}\begin{bmatrix} 6 & 3l_e & -6 & 3l_e \\ 3l_e & 2l_e^2 & -3l_e & l_e^2 \\ -6 & -3l_e & 6 & -3l_e \\ 3l_e & l_e^2 & -3l_e & 2l_e^2 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}_e = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}_e$$

<u>Set of linear equations for one element model of the considered cantilever beam:</u>

$$\frac{2EI}{l^3}\begin{bmatrix} 6 & 3l & -6 & 3l \\ 3l & 2l^2 & -3l & l^2 \\ -6 & -3l & 6 & -3l \\ 3l & l^2 & -3l & 2l^2 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \frac{p_0 l}{2} \\ \frac{-p_0 l^2}{12} \end{Bmatrix}$$

Constraints q_1=0 and q_2=0 may be taken into account by

the transformation of the set of equation to the form $[A]\begin{Bmatrix} F_1 \\ F_2 \\ q_3 \\ q_4 \end{Bmatrix} = \{b\}$ or by reductionof the problem to

$$\frac{2EI}{l^3}(6q_3 - 3lq_4) = \frac{p_0 l}{2},$$

$$\frac{2EI}{l^3}(-3lq_3 + 2l^2 q_4) = \frac{-p_0 l^2}{12},$$

Solution is:

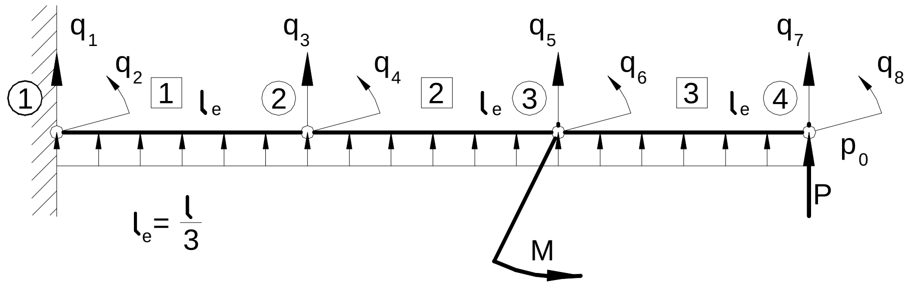
$$q_3 = \frac{1}{8}\frac{p_0 l^4}{EI}$$

$$q_4 = \frac{1}{6}\frac{p_0 l^3}{EI}$$

Finally the deflection function from the one element model is $w(\xi) = \left(\frac{3}{8} - \frac{1}{6}\right)\frac{p_0 l^2}{EI}\xi^2 + \left(\frac{-2}{8} + \frac{1}{6}\right)\frac{p_0 l}{EI}\xi^3 = \frac{5}{24}\frac{p_0 l^2}{EI}\xi^2 - \frac{p_0 l}{12EI}\xi^3$

The same result as obtained in the case of Ritz method – why?

Dividing the beam into LE elements

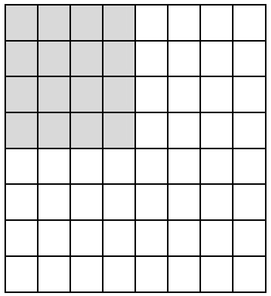


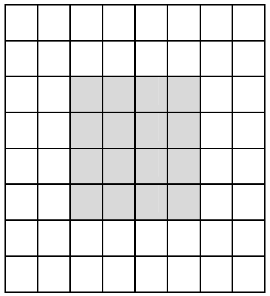
global nodal displacements vector $\{q\} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \\ w_3 \\ \theta_3 \\ w_4 \\ \theta_4 \end{Bmatrix}.$

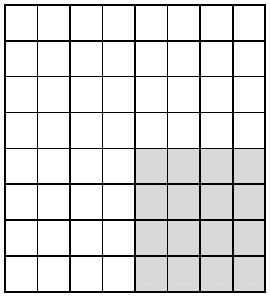
N=8 nodal diplacements (degrees of freedom of the FE model)

Strain energy U_e of each of the elements

$$U_e = \frac{1}{2} \underset{1x4}{\lfloor q \rfloor_e} \underset{4x4}{[k]_e} \underset{4x1}{\{q\}_e} \qquad = \frac{1}{2} \underset{1\times N}{\lfloor q \rfloor} \underset{N\times N}{\left[k^*\right]_e} \underset{N\times 1}{\{q\}},$$

$\left[k^*\right]_1 =$ 

$\left[k^*\right]_2 =$ 

$\left[k^*\right]_3 =$ 

element 1 with the global DOF :
q_1, q_2, q_3, q_4

element 2 with the global DOF :
q_3, q_4, q_5, q_6

element 3 with the global DOF :
q_5, q_6, q_7, q_8

$$U=\sum_{e=1}^{LE}U_e=\frac{1}{2}\lfloor q\rfloor\left(\sum_{i=1}^{LE}\left[k^*\right]_e\right)\{q\}=\frac{1}{2}\lfloor q\rfloor[K]\{q\}.$$

$$V=U-W_z=\frac{1}{2}\lfloor q\rfloor[K]\{q\}-\lfloor q\rfloor\{F\},$$

$$\frac{\partial V}{\partial q_i}=0,\qquad i=1,2,3,\ldots,n$$

$[K]\{q\}=\{F\}$. + displacement boundary conditions (constraints)

For each element the internal forces M,T are calculated separately:

$$M_q(\xi)=EIw''(\xi)=EI\left\lfloor N_1'',N_2'',N_3'',N_4''\right\rfloor\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}_e,$$

$$M_q(\xi)=\left[\frac{12}{l_e^3}(\xi-\frac{l_e}{2})q_1+\frac{6}{l_e^2}(\xi-\frac{2}{3}l_e)q_2-\frac{12}{l_e^3}(\xi-\frac{l_e}{2})q_3+\frac{6}{l_e^2}(\xi-\frac{l_e}{3})q_4\right]EI,$$

$$T(\xi)=-EIw'''(\xi)=EI\left\lfloor N_1''',N_2''',N_3''',N_4'''\right\rfloor\begin{Bmatrix}q_1\\q_2\\q_3\\q_4\end{Bmatrix}_e.$$

$$T(\xi)=-\left[\frac{12}{l_e^3}(q_1-q_3)+\frac{6}{l_e^2}(q_2+q_4)\right]EI.$$

For the case of 3-element model shown in the figure the final set of linear equations is

$$
\begin{bmatrix}
k_{11}^1 & k_{12}^1 & k_{13}^1 & k_{14}^1 & 0 & 0 & 0 & 0 \\
k_{21}^1 & k_{22}^1 & k_{23}^1 & k_{24}^1 & 0 & 0 & 0 & 0 \\
k_{31}^1 & k_{32}^1 & k_{33}^1 + k_{11}^2 & k_{34}^1 + k_{12}^2 & k_{13}^2 & k_{14}^2 & 0 & 0 \\
k_{41}^1 & k_{42}^1 & k_{43}^1 + k_{21}^2 & k_{44}^1 + k_{22}^2 & k_{23}^2 & k_{24}^2 & 0 & 0 \\
0 & 0 & k_{31}^2 & k_{32}^2 & k_{33}^2 + k_{11}^3 & k_{34}^2 + k_{12}^3 & k_{13}^3 & k_{14}^3 \\
0 & 0 & k_{41}^2 & k_{42}^2 & k_{43}^2 + k_{21}^3 & k_{44}^2 + k_{22}^3 & k_{23}^3 & k_{24}^3 \\
0 & 0 & 0 & 0 & k_{31}^3 & k_{32}^3 & k_{33}^3 & k_{34}^3 \\
0 & 0 & 0 & 0 & k_{41}^3 & k_{42}^3 & k_{43}^3 & k_{44}^3
\end{bmatrix}
\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \end{Bmatrix}
$$

$$
\frac{2EI}{l_e^3}
\begin{bmatrix}
6 & 3l_e & -6 & 3l_e & 0 & 0 & 0 & 0 \\
3l_e & 2l_e^2 & -3l_e & l_e^2 & 0 & 0 & 0 & 0 \\
-6 & -3l_e & 12 & 0 & -6 & 3l_e & 0 & 0 \\
3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 & 0 & 0 \\
0 & 0 & -6 & -3l_e & 12 & 0 & -6 & 3l_e \\
0 & 0 & 3l_e & l_e^2 & 0 & 4l_e^2 & -3l_e & l_e^2 \\
0 & 0 & 0 & 0 & -6 & -3l_e & 6 & -3l_e \\
0 & 0 & 0 & 0 & 3l_e & l_e^2 & -3l_e & 2l_e^2
\end{bmatrix}
\begin{Bmatrix} 0 \\ 0 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{Bmatrix}
=
\begin{Bmatrix} F_1 \\ F_2 \\ p_0 l_e \\ 0 \\ p_0 l_e \\ M \\ P + \dfrac{p_0 l_e}{2} \\ \dfrac{-p_0 l_e^2}{12} \end{Bmatrix}
$$

FEM calculations:

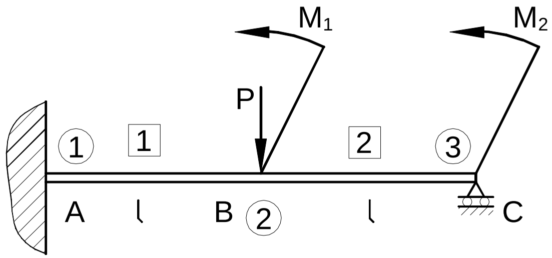
1. Generation of stiffness matrices $\underset{4x4}{[k]_e}$ for all elements
2. Assembling the element matrices to obtain the global stiffness matrix $\underset{NxN}{[K]}$
3. Finding the equivalent nodal force vector $\underset{Nx1}{\{F\}}$
4. Imposing the boundary conditions and the solution of the final set of linear equations – finding all nodal displacements $\underset{Nx1}{\{q\}}$
5. Calculation of the internal forces (bending moment, shear force) and the stresses within the elements

The example

Final set of equations (3 active DOF)

$$\frac{2EI}{l^3}\begin{bmatrix} 12 & 0 & 3l \\ 0 & 4l^2 & l^2 \\ 3l & l^2 & 2l^2 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}.$$

$$\begin{Bmatrix} q_3 \\ q_4 \\ q_6 \end{Bmatrix} = \begin{Bmatrix} w_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \frac{l}{96EI}\begin{bmatrix} 7l^2 & 3l & -12l \\ 3l & 15 & -12 \\ -12l & -12 & 48 \end{bmatrix}\begin{Bmatrix} -P \\ M_1 \\ M_2 \end{Bmatrix}.$$



(exact solution – why?)

5. BARS AND SPRINGS

Finite element of a bar under axial loads:

Assuming nodal displacements u_1 i u_2 we have $u(\xi)$ as the linear function: $u(\xi) = u_1 + \dfrac{u_2 - u_1}{l_e}\xi$.

After some operations $u(\xi)$ may be presented in the standard form as dependent on the nodal displacements and shape functions:

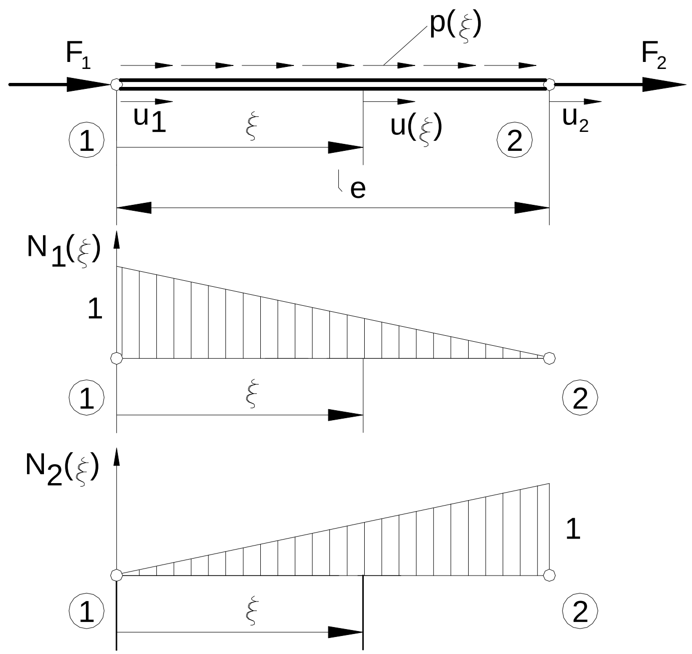
$$u(\xi) = \left(1 - \frac{\xi}{l}\right)u_1 + \frac{\xi}{l}u_2 = \lfloor N_1(\xi), N_2(\xi) \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \lfloor N \rfloor \{q\}_e ,$$

where

$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_e$ is the vector of nodal displacements

$\lfloor N \rfloor = \lfloor N_1(\xi), N_2(\xi) \rfloor$ is the vector of shape functions

$$N_1(\xi) = 1 - \frac{\xi}{l_e}, \qquad N_2(\xi) = \frac{\xi}{l_e},$$



Tension bar element with 2 nodes and 2 degrees of freedom and its shape functions

,

Strain energy of the element:

$$U_e = \frac{1}{2} A \int_0^{l_e} \sigma(\xi)\varepsilon(\xi) d\xi = \frac{EA}{2} \int_0^{l_e} (\varepsilon(\xi))^2 d\xi \,.$$

Taking into account that

$$\varepsilon(\xi) = \frac{du}{d\xi} = \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e .$$

we have

$$U_e = \frac{EA}{2} \int_0^{l_e} \lfloor q_1, q_2 \rfloor_e \begin{Bmatrix} N_1' \\ N_2' \end{Bmatrix} \lfloor N_1', N_2' \rfloor \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e d\xi =$$

$$= \frac{EA}{2} \lfloor q_1, q_2 \rfloor_e \int_0^{l_e} \begin{bmatrix} N_1'N_1' & N_1'N_2' \\ N_2'N_1' & N_2'N_2' \end{bmatrix} d\xi \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e ,$$

where

$$[k]_e = \frac{EA}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

is the stiffness matrix of the rod element (symmetric, singular, positive semidefinite)

Equivalent nodal forces

The forces equivalent to the distributed load $p(\xi)\left[\frac{\text{N}}{\text{m}}\right]$.

$$W_{ze}^{p}=\int_0^{l_e} p(\xi)u(\xi)d\xi=\int_0^{l_e}\lfloor N_1(\xi)\,p(\xi),N_2(\xi)\,p(\xi)\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e d\xi=$$

$$=\left\lfloor \int_0^{l_e} N_1(\xi)\,p(\xi),\int_0^{l_e} N_2(\xi)\,p(\xi)d\xi\right\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e .$$

In result:

$$W_{ze}^{p}=\lfloor F_1^e,F_2^e\rfloor_e\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e, \quad \text{where} \qquad F_i^e=\int_0^{l_e} N_i(\xi)p(\xi)d\xi\,,$$

F_i^e - the nodal forces equivalent to the distributed load p (‘work-equivalent’ or ‘kinematically’ equivalent)

Next steps of FE modelling are similar as in the case of the beam element. Finally we get the system of linear quations :

$$[K]\{q\}=\{F\}.$$

The right side vector $\{F\}$ contains the external forces acting on nodes of the model (active nodes and reactions).

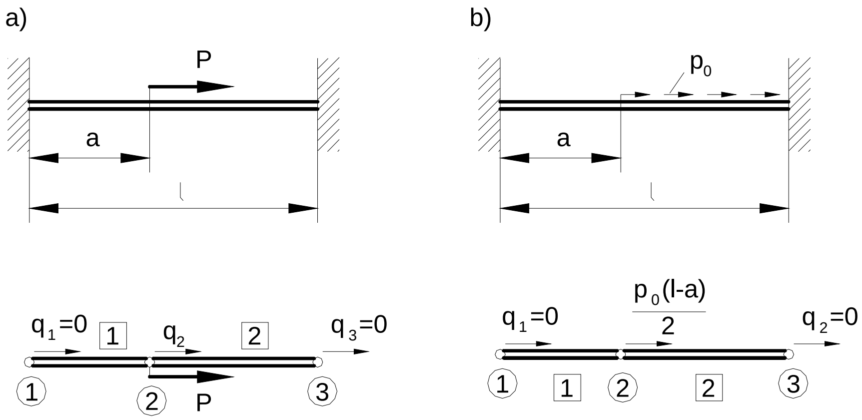
The system is solved after taking into account all boundary conditions;

When the vector of nodal displacements is determined the stresses within each of elements are computed:

$$\sigma=E\varepsilon=E\left\lfloor N_1'(\xi),N_2'(\xi)\right\rfloor\begin{Bmatrix}q_1\\q_2\end{Bmatrix}_e=\frac{E(q_2-q_1)}{l_e}.$$

Example.

Solve the presented below rods using FE models consisted of 2 elements



Stiffness matrices of the two finite elements

$$[k]_e^1 = \frac{EA}{a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^2 = \frac{EA}{l-a}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

System of simultaneous linear equations

$$EA\begin{bmatrix} \frac{1}{a} & -\frac{1}{a} & 0 \\ -\frac{1}{a} & \frac{1}{a}+\frac{1}{l-a} & -\frac{1}{l-a} \\ & -\frac{1}{l-a} & \frac{1}{l-a} \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

.

After including the boundary conditions $q_1 = q_3 = 0$ and $F_2 = P$ (<u>case a)</u>) we have

$$q_2 = \frac{P(l-a)a}{EAl},$$

$$F_1 = \frac{-P(l-a)}{l},$$

$$F_3 = \frac{-Pa}{l}.$$

where F_1 and F_3 are the nodal forces (reactions).

In the case b the nodal force in the second node is:

$$F_2 = \frac{p_0(l-a)}{2},$$

Thus $q_2 = \frac{p_0(l-a)^2 a}{2lEA}$, $F_1 = \frac{-p_0(l-a)^2}{2l}$, $F_3 = \frac{-p_0 a(l-a)}{2l}$.

The reaction in the first node $R_1 = F_1$

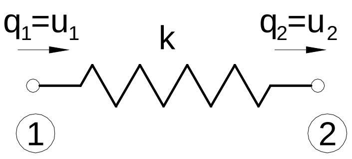
And the reaction in the third node

$$R_3 = F_3 - \frac{p_0(l-a)l}{2l} = \frac{-p_0 a(l-a)}{2l} - \frac{p_0(l-a)l}{2l} = \frac{-p_0(l-a)(l+a)}{2l}.$$

$$R_1 + R_3 = -p_0(l-a).$$

FE solution in the case a is the exact one but in the case b the approximate (why?)

Spring element



$F = k * \Delta u = k(u_2 - u_1)$

$[k]_e = k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

Finite element of a spring

Strain energy

$$U_e = \frac{1}{2} F \Delta u = \frac{1}{2} k (\Delta u)^2 = \frac{1}{2} k (u_2 - u_1)(u_2 - u_1).$$

$$U_e = \frac{1}{2} \lfloor u_1, u_2 \rfloor \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix},$$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e,$$

$$[k]_e = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix},$$ (stiffness matrix of a spring)

In the same way may be derived the stiffness matrix for the twisted shaft:

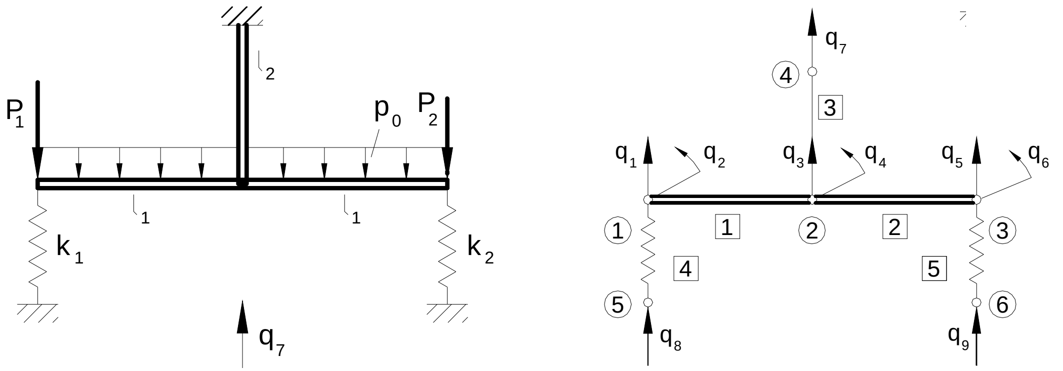
$$[k]_e = \frac{GI_s}{l_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},$$

where GI_s is a torsional stiffness and the nodal displacements correspond to the rotation of the end cross-sections.

The FE models of the elastic structures can be built dividing the structure into finite elements of different types (beams, tension bars, springs etc.)

Example:

Find the finie element system of equations $[K]\{q\}=\{F\}$ for the structure presented below



Solution
FE model may be created using 2 beam e e rod element and 2 spring elements. The total number of degrees of freedom is 9
The stiffness matrices of the beam elements

$$[k]_e^1=[k]_e^2=\frac{2EI}{l_1^3}\begin{bmatrix} 6 & 3l_1 & -6 & 3l_1 \\ 3l_1 & 2l_1^2 & -3l_1 & l_1^2 \\ -6 & -3l_1 & 6 & -3l_1^2 \\ 3l_1 & l_1^2 & -3l_1 & 2l_1^2 \end{bmatrix}.$$

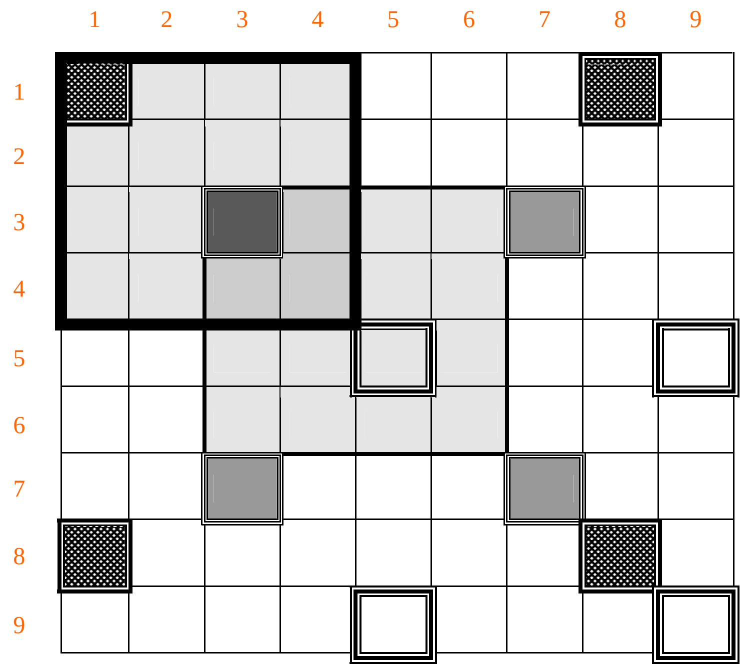
Degrees of freedom of the first element are q_1, q_2, q_3, q_4, and for the second q_3, q_4, q_5, q_6.
The stiffness matrix of the rod element (with the degrees of fredom q_3 and q_7). is

$$[k]_e^3=\frac{EA}{l_2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

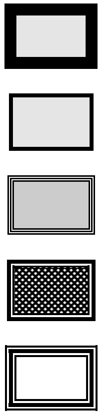
The stiffness matrices of the springs:

$$[k]_e^4=k_1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad [k]_e^5=k_2\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ and corresponding degrees of freedom are q_8, q_1 and $q_9\ q_5$.

The FE system of equations $[K]\{q\}=\{F\}$ for the assuming numbering of the degrees of freedom:



$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{-p_0 l_1}{2} - P_1 \\ \dfrac{-p_0 l_1^2}{12} \\ p_0 l_1 \\ 0 \\ \dfrac{-p_0 l_1}{2} - P_2 \\ \dfrac{p_0 l_1^2}{12} \\ F_7 \\ F_8 \\ F_9 \end{Bmatrix}$$



– Coefficients of the stiffness matrix of the element No 1 (beam)

– Coefficients of the stiffness matrix of the element No 2 (beam)

– Coefficients of the stiffness matrix of the element No 3 (rod)

– Coefficients of the stiffness matrix of the element No 4 (spring)

– Coefficients of the stiffness matrix of the element No 5 (spring)

$[K]$ may be written in the form

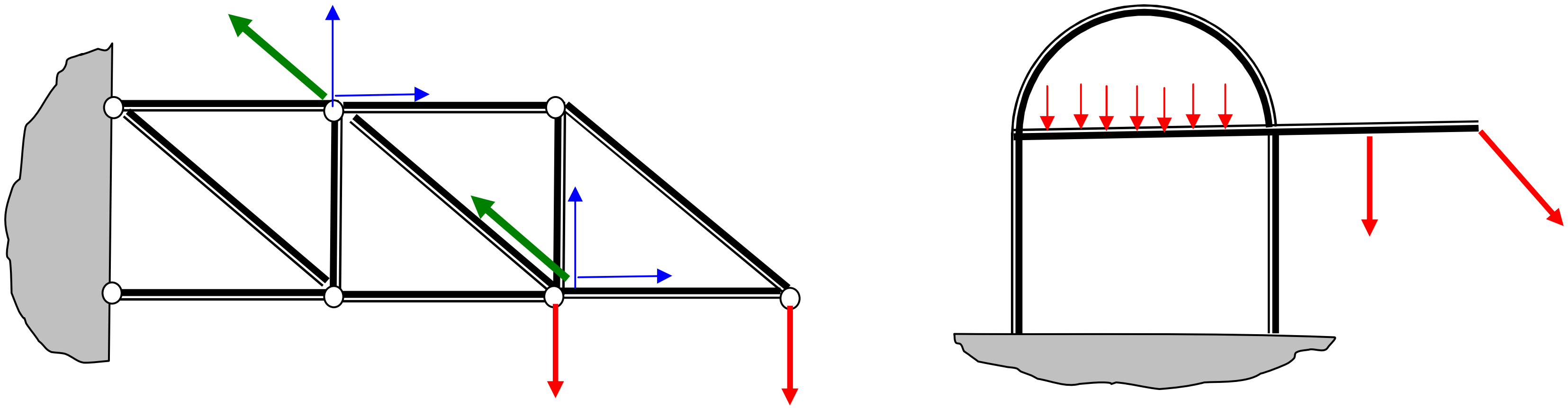
$\underset{9\times9}{[K]} =$

$k_{11}^{1}+k_{22}^{4}$	k_{12}^{1}	k_{13}^{1}	k_{14}^{1}	0	0	0	k_{12}^{4}	0
k_{21}^{1}	k_{22}^{1}	k_{23}^{1}	k_{24}^{1}	0	0	0	0	0
k_{31}^{1}	k_{32}^{1}	$k_{33}^{1}+k_{11}^{2}+k_{11}^{3}$	$k_{34}^{1}+k_{12}^{2}$	k_{13}^{2}	k_{14}^{2}	k_{12}^{3}	0	0
k_{41}^{1}	k_{42}^{1}	$k_{43}^{1}+k_{21}^{2}$	$k_{44}^{1}+k_{22}^{2}$	k_{23}^{2}	k_{24}^{2}	0	0	0
0	0	k_{31}^{2}	k_{32}^{2}	$k_{33}^{2}+k_{22}^{5}$	k_{34}^{2}	0	0	k_{12}^{5}
0	0	k_{41}^{2}	k_{42}^{2}	k_{43}^{2}	k_{44}^{2}	0	0	0
0	0	k_{21}^{3}	0	0	0	k_{11}^{3}	0	0
k_{21}^{4}	0	0	0	0	0	0	k_{11}^{4}	0
0	0	0	0	k_{21}^{5}	0	0	0	k_{11}^{5}

6. TRUSSES AND FRAMES

Trusses - structures made of simple straight bars (members), joined at their ends (nodes).

External forces and reactions to those forces are considered to act only at the nodes and result in forces in the members which are either tensile or compressive forces. Other internal forces are explicitly excluded because all the joints in a truss are treated as articulated joints.



The examples of 2D truss and 2D frame

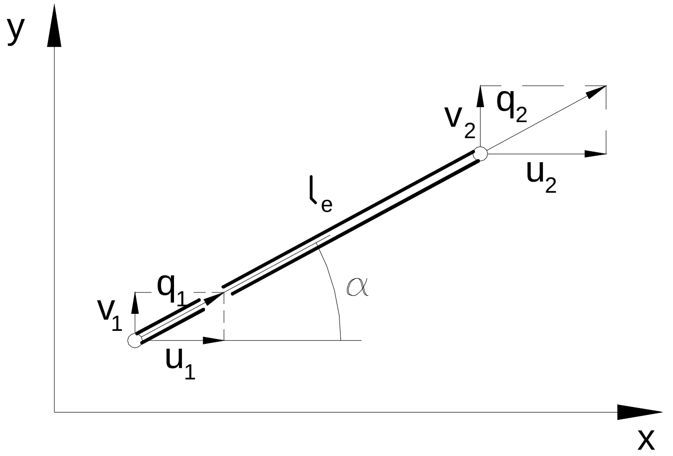
Frames are the structures with members that are rigidly connected - e.g. with welded joints. The members of frames can be loaded by concentrated and distributed forces. As a result they carry all possible internal forces (normal and shear forces, bending moments and torsinal moments).

TRUSSES

2D trusses

Relation between the nodal displacements in local (element) coordinate systems and in global coordinates

$\lfloor q \rfloor_e = \lfloor q_1, q_2 \rfloor_e$ along the axis of the rod $\lfloor q_g \rfloor_e = \lfloor u_1, v_1, u_2, v_2 \rfloor_e$ in x,y coordinate system



Finite element of a plane truss

$$q_i = u_i \cos\alpha + v_1 \sin\alpha \,. \quad (i = 1,2)$$

$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}_e = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 & 0 \\ 0 & 0 & \cos\alpha & \sin\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}_e ,$$

$$\{q\}_e = [T_k]\{q_q\}_e$$

Strain energy of the element

$$U_e = \frac{1}{2} \underset{1\times2}{\lfloor q \rfloor} \underset{2\times2}{[k]_e} \underset{2\times1}{\{q\}_e} = \frac{1}{2} \underset{1\times4}{\lfloor q_q \rfloor_e^T} \underset{4\times2}{[T_k]^T} \underset{2\times2}{[k]_e} \underset{2\times4}{[T_k]} \underset{4\times1}{\{q_q\}_e}$$

$$U_e = \frac{1}{2} \lfloor q_q \rfloor_e [k_g]_e \{q_q\}_e ,$$

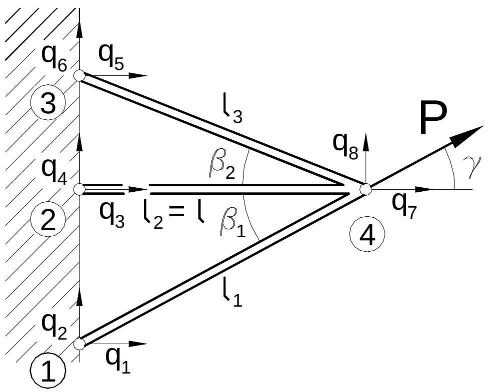
$$[k_g]_e = \frac{EA}{l_e} \begin{bmatrix} c^2 & sc & -c^2 & -sc \\ sc & s^2 & -sc & -s^2 \\ -c^2 & -sc & c^2 & sc \\ -sc & -s^2 & sc & s^2 \end{bmatrix} \qquad (*)$$

$$s = \sin\alpha, \; c = \cos\alpha$$

The stiffness matrix of the truss element in global coordinate system

Example.

Find the displacement vector of the node 4 of the simple 2D truss for the case $\beta_1 = \beta_2$ and the horizontal force P $(\gamma = 0)$.



Rozwiązanie

Element 1 nodes 1 and 4 slope angle $\alpha_1 = \beta_1$ length $l_1 = \dfrac{l}{\cos\alpha_1}$.

Element 2 nodes 2 and 4 slope angle $\alpha_2 = 0$ length $l_2 = \dfrac{l}{\cos\alpha_2}$.

Element 3 nodes 3 and 4 slope angle $\alpha_3 = -\beta_2$ length $l_3 = \dfrac{l}{\cos\alpha_3}$.

.The stiffness matrices of the three elements $\left[k_{ij}\right]_e^1, \left[k_{ij}\right]_e^2, \left[k_{ij}\right]_e^3$ are defined by (*).

The system of FE equations:

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 & 0 & 0 & 0 & k_{13}^1 & k_{14}^1 \\ k_{21}^1 & k_{22}^1 & 0 & 0 & 0 & 0 & k_{23}^1 & k_{24}^1 \\ 0 & 0 & k_{11}^2 & k_{12}^2 & 0 & 0 & k_{13}^2 & k_{14}^2 \\ 0 & 0 & k_{21}^2 & k_{22}^2 & 0 & 0 & k_{23}^2 & k_{24}^2 \\ 0 & 0 & 0 & 0 & k_{11}^3 & k_{12}^3 & k_{13}^3 & k_{14}^3 \\ 0 & 0 & 0 & 0 & k_{21}^3 & k_{22}^3 & k_{23}^3 & k_{24}^3 \\ k_{31}^1 & k_{32}^1 & k_{31}^2 & k_{32}^2 & k_{31}^3 & k_{32}^3 & k_{33}^1 + k_{33}^2 + k_{33}^3 & k_{34}^1 + k_{34}^2 + k_{34}^3 \\ k_{41}^1 & k_{42}^1 & k_{41}^2 & k_{42}^2 & k_{41}^3 & k_{42}^3 & k_{43}^1 + k_{43}^2 + k_{43}^3 & k_{44}^1 + k_{44}^2 + k_{44}^3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ P\cos\gamma \\ P\sin\gamma \end{Bmatrix}.$$

Taking into account that $q_j = 0$ for $j = 1,6$ the set of quations may be reduced to

$$EA\left[\begin{array}{c|c} \sum_{i=1}^{3}\frac{c_i^2}{l_i} & \sum_{i=1}^{3}\frac{s_i c_i}{l_i} \\ \hline \sum_{i=1}^{3}\frac{s_i c_i}{l_i} & \sum_{i=1}^{3}\frac{s_i^2}{l_i} \end{array}\right]\begin{Bmatrix} q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} P\sin\gamma \\ P\cos\gamma \end{Bmatrix}.$$

Assuming $\beta_1 = \beta_2 = \beta$ $\gamma = 0$

$$\frac{EA}{l}\left[\begin{array}{c|c} 1+2c^3 & 0 \\ \hline 0 & 2s^2c \end{array}\right]\begin{Bmatrix} q_7 \\ q_8 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix},$$

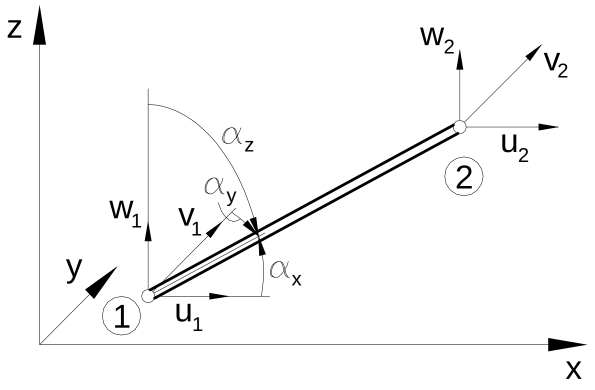
where $c = \cos\beta$, $s = \sin\beta$.
Then

$$q_7 = \frac{Pl}{EA(1+2c^3)},$$

$$q_8 = 0.$$

The normal forces in the elements are calculated from the nodal displacements in the local (element) coordinate systems

3D truss element in the coordinate system x,y,z



$$\{q\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{Bmatrix}$$

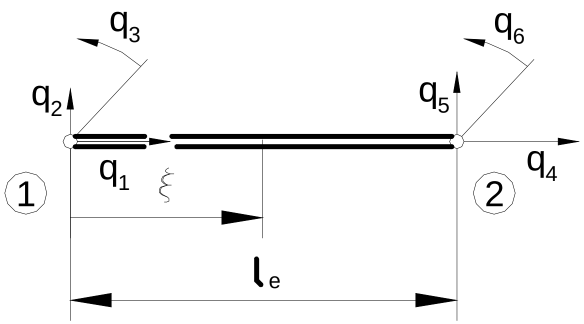
$$\left[k^g\right]_e = \frac{EA}{l_e} \begin{bmatrix} c_x^2 & c_x c_y & c_x c_z & -c_x^2 & -c_x c_y & -c_x c_z \\ c_x c_y & c_y^2 & c_y c_z & -c_x c_y & -c_y^2 & -c_y c_z \\ c_x c_z & c_y c_z & c_z^2 & -c_x c_z & -c_y c_z & -c_z^2 \\ -c_x^2 & -c_x c_y & -c_x c_z & c_x^2 & c_x c_y & c_x c_z \\ -c_x c_y & -c_y^2 & -c_y c_z & c_x c_y & c_y^2 & c_y c_z \\ -c_x c_z & -c_y c_z & -c_z^2 & c_x c_z & c_y c_z & c_z^2 \end{bmatrix}$$

where $c_x = \cos\alpha_x$, $c_y = \cos\alpha_y$, $c_z = \cos\alpha_z$.

FRAMES

2D frame element in the local coordinate system

The stiffness matrix of a frame element assembled from the stiffness matrices of the beam element with four degrees of freedom and the rod element with 2 degrees of freedom:



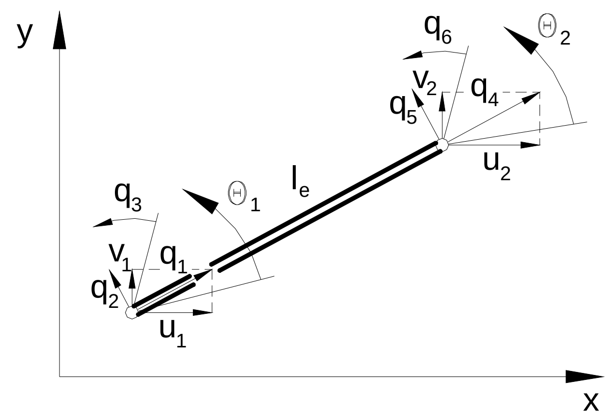
$$
\begin{matrix} & 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}
$$

$$
[k]_e = \begin{bmatrix}
\frac{EA}{l_e} & 0 & 0 & -\frac{EA}{l_e} & 0 & 0 \\
0 & \frac{12EI}{l_e^3} & \frac{6EI}{l_e^2} & 0 & \frac{-12EI}{l_e^3} & \frac{6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{4EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{2EI}{l_e} \\
\frac{-EA}{l_e} & 0 & 0 & \frac{EA}{l_e} & 0 & 0 \\
0 & \frac{-12EI}{l_e^3} & \frac{-6EI}{l_e^2} & 0 & \frac{12EI}{l_e^3} & \frac{-6EI}{l_e^2} \\
0 & \frac{6EI}{l_e^2} & \frac{2EI}{l_e} & 0 & \frac{-6EI}{l_e^2} & \frac{4EI}{l_e}
\end{bmatrix}
$$

The stiffness matrix of a frame element in local coordinate system

The element with 6 DOF , the deformation defined by the functions $u(\xi)$ i $w(\xi)$ in the local c.s. It is called also 2D beam element.

2D frame element in the global coordinate system xy



$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \qquad \{q_g\}_e = \begin{Bmatrix} u_1 \\ v_1 \\ \Theta_1 \\ u_2 \\ v_2 \\ \Theta_2 \end{Bmatrix}$$

The vectors of DOF of the frame element in the local c.s. $\{q\}_e$ and in the global c.s. $\{q_g\}_e$

The relation between the displacement of a node 1 in local (element) coordinate system and in global coordinate system

$$\begin{cases} q_1 = u_1 \cos\alpha + \upsilon_1 \sin\alpha, \\ q_2 = -u_1 \sin\alpha + \upsilon_1 \cos\alpha, \\ q_3 = \theta_1. \end{cases}$$

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix}_e = [T_r] \begin{Bmatrix} u_1 \\ \upsilon_1 \\ \theta_1 \\ u_2 \\ \upsilon_2 \\ \theta_2 \end{Bmatrix} = [T_r] \cdot \{q_g\}_e ,$$

where the transformation matrix $[T_r]$ is

$$[T_r] = \left[\begin{array}{c|c|c|c|c|c} c & s & 0 & 0 & 0 & 0 \\ \hline -s & c & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & c & s & 0 \\ \hline 0 & 0 & 0 & -s & c & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Strain energy of the element

$$U_e = \frac{1}{2}\lfloor q \rfloor_e [k]_e \{q\}_e = \frac{1}{2}\lfloor q_g \rfloor_e [T_r]^T [k]_e [T_r]\{q_g\}_e,$$

$$U_e = \frac{1}{2}\lfloor q_g \rfloor_e \left[k^g\right]_e \{q_g\}_e,$$

where $$\left[k^g\right]_e = [T_r]^T [k]_e [T],$$

is the stiffness matrix of the 3D frame element in global c.s.

3D frames (beams)

The local (element) coordinate system is connected with the axis of the element.

The element x’is oriented along the element.

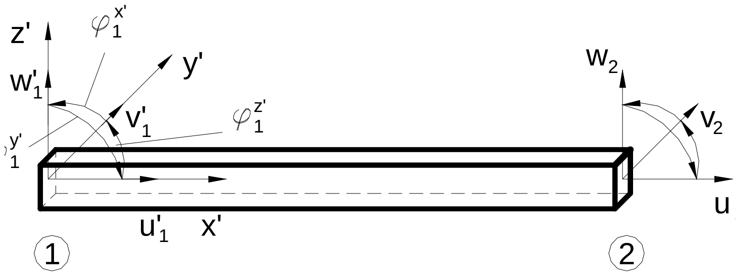
The y’ axis is automatically set parallel to the global xy plane

(If the element is perpendicular to the xy plane the x’ is located

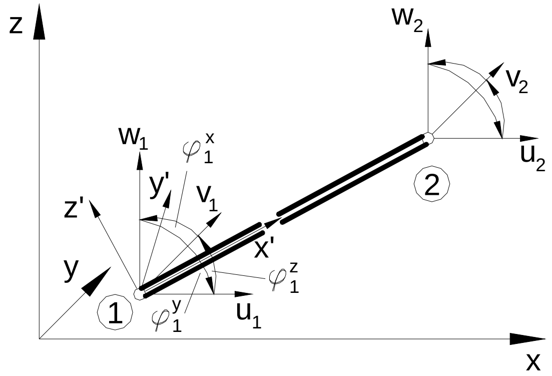
in parralel to the global y axis)

The element input data include:

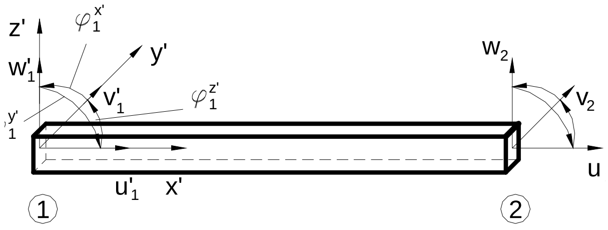
- the node locations
- the cross-sectional area
- 2 moments of inertia about the principal axes of the section
- the parameters defining shear stiffness in the principal directions and the torsional stiffness



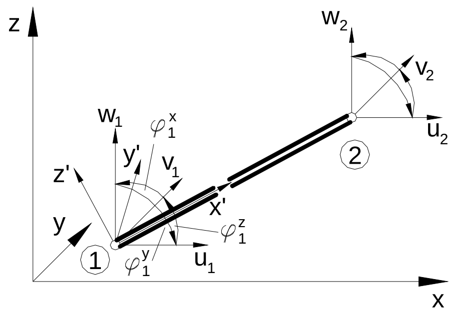
$$\{q\}_e = \lfloor u'_1, v'_1, w'_1, \varphi^{x'}_1, \varphi^{y'}_1, \varphi^{z'}_1, u'_2, v'_2, w'_2, \varphi^{x'}_2, \varphi^{y'}_2, \varphi^{z'}_2 \rfloor^T$$



$$\{q\}_e = \lfloor u_1, v_1, w_1, \varphi^{x}_1, \varphi^{y}_1, \varphi^{z}_1, u_2, v_2, w_2, \varphi^{x}_2, \varphi^{y}_2, \varphi^{z}_2 \rfloor^T$$



$$\{q\}_e = \lfloor u'_1, v'_1, w'_1, \varphi_1^{x'}, \varphi_1^{y'}, \varphi_1^{z'}, u'_2, v'_2, w'_2, \varphi_2^{x'}, \varphi_2^{y'}, \varphi_2^{z'} \rfloor^T$$



$$\{q\}_e = \lfloor u_1, v_1, w_1, \varphi_1^{x}, \varphi_1^{y}, \varphi_1^{z}, u_2, v_2, w_2, \varphi_2^{x}, \varphi_2^{y}, \varphi_2^{z} \rfloor^T$$

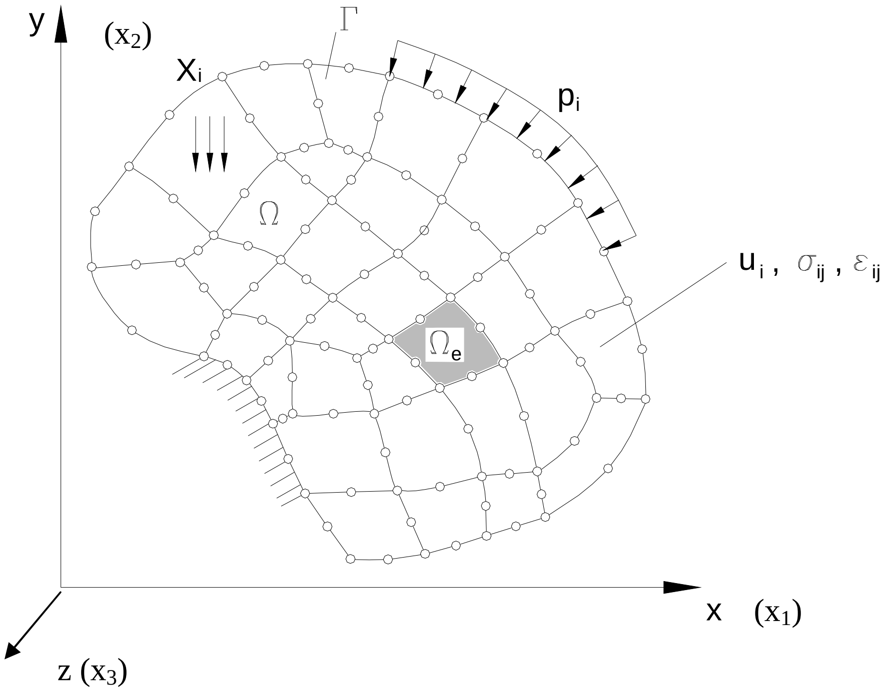
$$[k]_e = \begin{bmatrix}
\frac{EA}{l_e} & & & & & & & & & & & \text{macierz symetryczna} \\
0 & \frac{12EI_{z'}}{l_e^3} & & & & & & & & & & \\
0 & 0 & \frac{12EI_{y'}}{l_e^3} & & & & & & & & & \\
0 & 0 & 0 & \frac{GI_s}{l_e} & & & & & & & & \\
0 & 0 & -\frac{6EI_{y'}}{l_e^2} & 0 & \frac{4EI_{y'}}{l_e} & & & & & & & \\
0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{4EI_{z'}}{l_e} & & & & & & \\
-\frac{EA}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{EA}{l_e} & & & & & \\
0 & -\frac{12EI_{z'}}{l_e^3} & 0 & 0 & 0 & -\frac{6EI_{z'}}{l_e^2} & 0 & \frac{12EI_{z'}}{l_e^3} & & & & \\
0 & 0 & -\frac{12EI_{y'}}{l_e^3} & 0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{12EI_{y'}}{l_e^3} & & & \\
0 & 0 & 0 & -\frac{GI_s}{l_e} & 0 & 0 & 0 & 0 & 0 & \frac{GI_s}{l_e} & & \\
0 & 0 & -\frac{6EI_{y'}}{l_e^2} & 0 & \frac{12EI_{z'}}{l} & 0 & 0 & 0 & \frac{6EI_{y'}}{l_e^2} & 0 & \frac{4EI_{y'}}{l_e} & \\
0 & \frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{2EI_{z'}}{l_e^2} & 0 & -\frac{6EI_{z'}}{l_e^2} & 0 & 0 & 0 & \frac{4EI_{z'}}{l_e}
\end{bmatrix}$$

3D beam element and the corresponding stiffness matrix in the local (element) coordinate system $x'y'z'$

7. TWO AND THREE- DIMENSIONAL LINEAR ELASTOSTATICS

The finite elements of trusses and beams are, due to specific assumptions and simplifications, one –dimensional. All field problems of stress analysis are in fact three-dimensional. In some limited cases the mathematical description of the problem may be formally reduced to two dimensional models (plane stress state, plane strain state, axisymmetry) or ore even one dimensional as discussed bef.

Consider a linearly elastic body of volume *Ω*, which is bounded by surface *Γ*.



Data:

Ω –the analysed volume (domain),

Γ –the boundary,

p_i –boundary tractions [N/m²].,

X_i –body forces [N/m³].

prescribed displacements u_i on on the part of the boundary Γ

Unknown internal fields:

u_i –displacement field,

ε_{ij} – strain state tensor,

σ_{ij} – strss state tensor,

The body is referred to a three (or two) dimensional, rectangular, right-handed Cartesian coordinate system x_i , i=1,3 (or x,y,z). The body is in static equilibrium under the action of body forces X_i in *Ω*, *prescribed surface tractions* p_i and *prescribed displacements* u_i on on the boundary Γ The three unknown internal fields are *displacements* u_i , *strains* ε_{ij} and *stresses* σ_{ij} . All of them are defined in *Ω*.

Component notation (Einstein indicial notation) for Cartesian tensors

The notation is used in rectangular Cartesian coordinates. In this notation, writing u_i is equivalent to writing the three components u_1, u_2, u_3 of the displacement field **u**.

The Einstein summation convention is a tensor notation, which is commonly used to implicitly define a sum. The convention states that when an index is repeated in a term that implies a sum over all possible values for that index.

Three examples:

$$\frac{\partial u_i}{\partial x_i} = \sum_i \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$\frac{\partial u_i}{\partial x_j} n_j = \sum_j \frac{\partial u_i}{\partial x_j} n_j = \frac{\partial u_i}{\partial x_1} n_1 + \frac{\partial u_i}{\partial x_2} n_2 + \frac{\partial u_i}{\partial x_3} n_3$$

$a_{ij}\, x_j = b_i$ i,j=1,n denotes the set of n linear equations

The indication of derivatives of tensors is simply illustrated in indicial notation by a comma.

$$f_{,i} = \frac{\partial f}{\partial x_i}$$

The comma in the above indicial notation indicates to take the derivative of f with respect to the coordinate x_i .

Examples: $u_{i,i} = \frac{\partial u_i}{\partial x_i} = \sum_i \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

$$u_{i,j} n_j = \frac{\partial u_i}{\partial x_j} n_j = \sum_j \frac{\partial u_i}{\partial x_j} n_j = \frac{\partial u_i}{\partial x_1} n_1 + \frac{\partial u_i}{\partial x_2} n_2 + \frac{\partial u_i}{\partial x_3} n_3$$

The Kronecker delta is a convenient way of expressing the identity in indicial notation: $\delta_{ij} = \begin{Bmatrix} 1 & if\ i = j \\ 0 & if \neq j \end{Bmatrix}$

The Kronecker delta follows the rules of index notation: $A_{ik} = \delta_{ij} A_{jk}$

Strain state:

3 extensional strains

$$\varepsilon_x = \frac{\partial u_x}{\partial x}$$

$$\varepsilon_y = \frac{\partial u_y}{\partial y}$$

$$\varepsilon_z = \frac{\partial u_z}{\partial z}$$

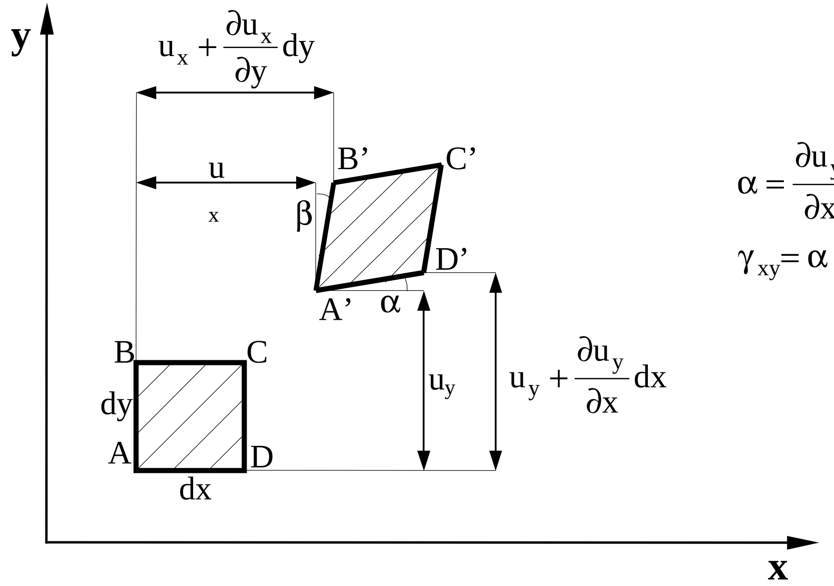
3 shearing strains

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$

$$\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$

$$\gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}$$

γ_{xy} , γ_{yz}, γ_{zx} - engineering shearing strains



$$\alpha = \frac{\partial u_y}{\partial x} \qquad \beta = \frac{\partial u_x}{\partial y}$$

$$\gamma_{xy} = \alpha + \beta$$

The strains may be written in the form of symmetric matrix assuming that

$\varepsilon_{xy} = \gamma_{xy}/2$, $\varepsilon_{yz} = \gamma_{yz}/2$, $\varepsilon_{zx} = \gamma_{xz}/2$. In this case the strains components form the symmetrical strain tensor.

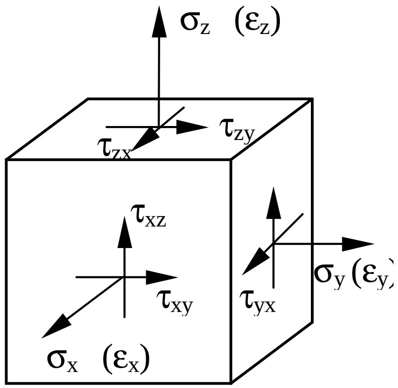
The components of the strain tensor ε_{ij} are often written in the form of symmetric matrix.

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{Bmatrix}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (\varepsilon_{ij} = \varepsilon_{ji}) \qquad \text{- kinematic equations}$$

Stress state : stress tensor σ_{ij}

Constitutive equations (3D Hook's law)



$\sigma_x = \sigma_{xx}$ $\sigma_y = \sigma_{yy}$ $\sigma_z = \sigma_{zz}$

$$\left.\begin{aligned}
\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\
\varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \\
\varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] \\
\gamma_{xy} &= \frac{1}{G}\tau_{xy} \\
\gamma_{yz} &= \frac{1}{G}\tau_{yz} \\
\gamma_{xz} &= \frac{1}{G}\tau_{xz}
\end{aligned}\right\} \Rightarrow
\begin{aligned}
\sigma_x &= \frac{E}{1+\nu}\left[\varepsilon_x + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
\sigma_y &= \frac{E}{1+\nu}\left[\varepsilon_y + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
\sigma_z &= \frac{E}{1+\nu}\left[\varepsilon_z + \frac{\nu}{1-2\nu}(\varepsilon_x + \varepsilon_y + \varepsilon_z)\right] \\
\tau_{xy} &= G\cdot\gamma_{xy} \\
\tau_{yz} &= G\cdot\gamma_{yz} \\
\tau_{xz} &= G\cdot\gamma_{xz}
\end{aligned}$$

E-Young's modulus, $\mathbf{G = \frac{E}{2(1+\nu)}}$ - shear modulus, ν- Poisson's ratio

$$\sigma_{ij} = 2G\left[\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}(\varepsilon_{kk})\right] \qquad \varepsilon_{ij} = \frac{1}{2G}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\delta_{ij}\sigma_{kk}\right) \qquad (\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Strain energy density:

$$U' = \frac{1}{2}\left[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right]$$

$$U' = \tfrac{1}{2}\ \varepsilon_{ij}\ \sigma_{ij}$$

Principle of the total potential energy:

$$V = U - W_z = \frac{1}{2}\int_\Omega \sigma_{ij}\varepsilon_{ij}\,d\Omega - \int_\Omega X_i u_i\,d\Omega - \int_\Gamma p_i u_i\,d\Gamma = \min,$$

Matrix notation

Matrix notation is a modification of direct tensor notation in which everything is placed in matrix form, with some trickery used if need be. The main advantages of the matrix notation are historical compatibility with finite element formulations, and ready computer implementation in symbolic or numeric form.

The representation of scalars, which may be viewed as 1 × 1 matrices, does not change. Neither does the representation of vectors because vectors are column (or row) matrices. Two-dimensional *symmetric* tensors are converted to one-dimensional arrays that list only the independent components (six in three dimensions, three in two dimensions). Component order is a matter of convention, but usually the diagonal components are listed first followed by the off-diagonal components.

For the strain and stress tensors this "vectorization" process produces the vectors $\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}, \quad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix},$

The relation between the strains and the displacement components in matrix notation:

$$\{\, \varepsilon(x,y,z) \,\} = [R]\{\, u(x,y,z) \,\},$$

$[R]$ is called symmetric gradient matrix in the continuum mechanics literature.

For 3 dimensional case :

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}, \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}, \qquad [R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}, \qquad \{u\} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix} = \begin{Bmatrix} u \\ \upsilon \\ w \end{Bmatrix}$$

In 2D case

$$\sigma = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}, \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \qquad [R] = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}, \qquad \{u\} = \begin{Bmatrix} u \\ \upsilon \end{Bmatrix}.$$

Hook's law:

$$\{\sigma\}=[D]\{\varepsilon\}, \qquad [D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2v}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2v}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2v}{2} \end{bmatrix}.$$

<u>Plane stress state</u> $\left(\sigma_z=0,\ \tau_{yz}=0,\ \tau_{zx}=0\right)$

$$[D]=\frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix}$$

<u>Plane strain state</u> $\left(\varepsilon_z=0,\ \gamma_{yz}=0,\ \gamma_{zx}=0\right)$

$$[D]=\frac{E}{(1+v)(1-2v)}\begin{bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1-2v}{2} \end{bmatrix}.$$

Strain energy density

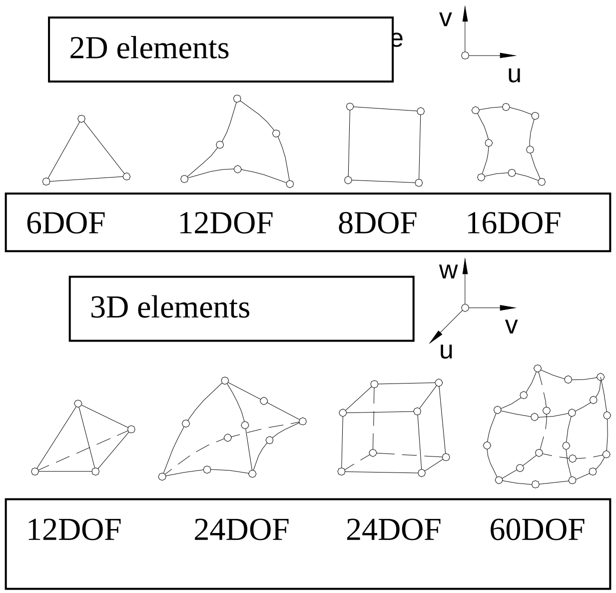
$$U^{'}=\frac{1}{2}\lfloor\varepsilon\rfloor\{\sigma\}$$

Total potential energy :

$$V=U-W_z=\frac{1}{2}\int_{\Omega}\lfloor\varepsilon\rfloor\{\sigma\}d\Omega-\int_{\Omega}\lfloor X\rfloor\{u\}d\Omega-\int_{\Gamma}\lfloor p\rfloor\{u\}\,d\Gamma$$

Finite element method for 2D and 3D problems of theory of elasticity:

The domain Ω is divided into the subdomains (finite elements) Ω_i : $\Omega = \bigcup_{i=1}^{LE} \Omega_i \qquad \Omega_i \cap \Omega_j = 0 \qquad i \neq j$.



2D and 3D finite elements

Displacement field over the element is interpolated from the nodal displacements:

$$\{u\} = [N(x, y, z)]\, \{q\}_e ,$$

where $\{q\}_e$ - nodal displacements vector , $[N]$ - shape functions matrix.

For example for the simplest trangular element with 3 nodes and 6 DOF the relation is

$$\begin{Bmatrix} u(x,y) \\ \upsilon(x,y) \end{Bmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \begin{Bmatrix} u_1 \\ \upsilon_1 \\ u_2 \\ \upsilon_2 \\ u_3 \\ \upsilon_3 \end{Bmatrix}$$ where N_i are the linear functions

Shape functions N_{ij} are usually polynomials defined in local (element) coordinate systems.

Displacements, strains and stresses within each element are defined as the functions of the coordinates of the considered point and the nodal displacements

$$\{u\} = [N]\{q\}_e ,$$
$$\{\varepsilon\} = [R]\{u\} = [R][N]\{q\}_e = [B]\{q\}_e ,$$ [B] – strain-displacement matrix
$$\{\sigma\} = [D]\{\varepsilon\} = [D][B]\{q\}_e .$$

The strain energy of the element Ω_e is:

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor \varepsilon \rfloor \{\sigma\}\, d\Omega_e .$$

$$U_e = \frac{1}{2}\int_{\Omega_e} \lfloor q \rfloor_e [B]^T [D]\, [B]\, \{q\}_e \, d\Omega_e , \qquad U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e .$$

Where

$$[k]_e = \int_{\Omega_e} [B]^T [D][B] d\Omega_e = \int_{\Omega_e} \left[B^*\right] d\Omega_e ,$$

is called **the stiffness matrix of the element** (symmetrical, singular, semi-positive defined) with the range equal to the number of DOF of the element. Matrix [B] depends on the position within the element so the integration requires the special numerical techniques.

Total strain energy of the structure is the sum of the finite elements energy:

$$U = \sum_{e=1}^{LE} U_e \,. \quad \text{(LE- number of finite elements in the model)}$$

Using the global nodal displacement vector {q}

$$U = \frac{1}{2} \underset{1\times n}{\lfloor q \rfloor} \underset{n\times n}{[K]} \underset{nx1}{\{q\}},$$

where n is total number of DOF of the model and $[K]$ is the stiffness matrix of the model.

The next step in FEM algorithms is finding the equivalent nodal forces {F} corresponding to the distributet loads {p} and {X}.

The total potential energy of the model is:

$$V = U - W_z = \frac{1}{2} \underset{1\times n}{\lfloor q \rfloor} \underset{n\times n}{[K]} \underset{n\times 1}{\{q\}} - \underset{1\times n}{\lfloor q \rfloor} \underset{n\times 1}{\{F\}},$$

The minimum is determined by the conditions

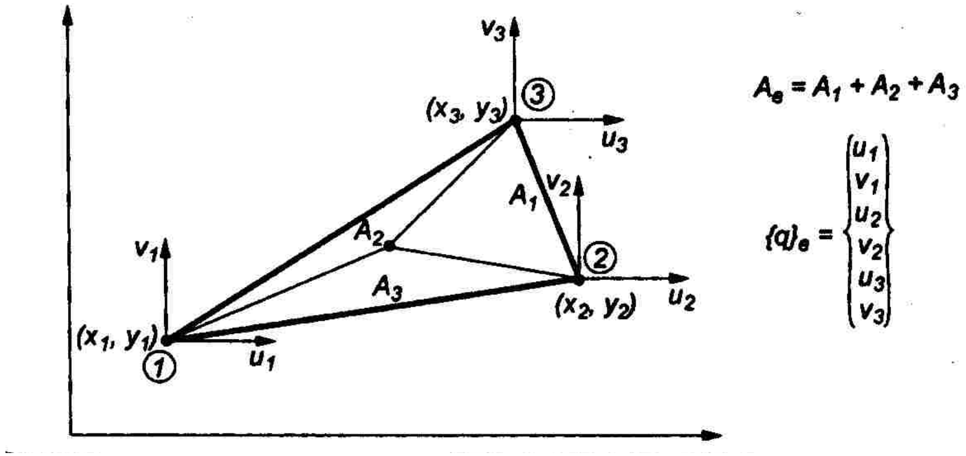
$$\frac{\partial V}{\partial q_i} = 0,$$

$$[K]\{q\} = \{F\}. \qquad \text{(to be solved using neccesary displacement boundary conditions)}$$

The strain and stress components in each finite element are found using the relations

$$\{\varepsilon\} = [B]\{q\}_e, \quad \{\sigma\} = [D]\{\varepsilon\} = [D][B]\{q\}_e$$

8. CONSTANT STRAIN TRIANGLE (CST)



$$u(x,y) = \sum_{i=1}^{3} N_i(x,y) \cdot u_i$$

$$v(x,y) = \sum_{i=1}^{3} N_i(x,y) \cdot v_i$$

$N_i(x_i, y_i) = 1, \qquad N_i(x_j, y_j) = 0 \quad \text{for } i \neq j$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) & 0 \\ 0 & N_1(x,y) & 0 & N_2(x,y) & 0 & N_3(x,y) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}_e$$

$$\{u\} = [N]\{q\}_e$$

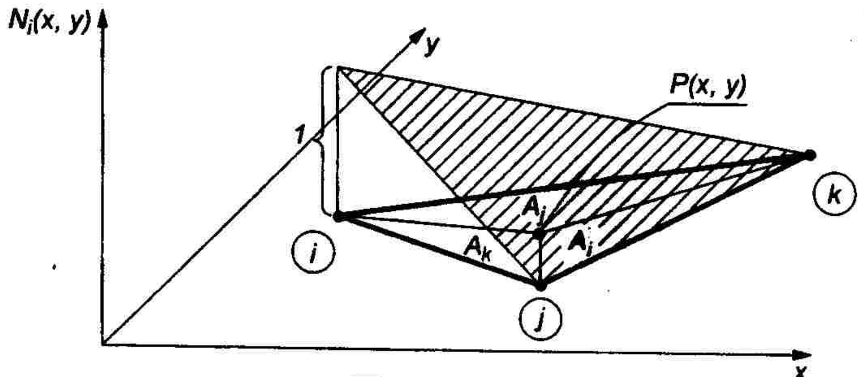
$$N_i(x,y)=\frac{A_i(x,y)}{A_e}\qquad A_e=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x_1 & x_2 & x_3\\ y_1 & y_2 & y_3\end{vmatrix}\qquad A_2=\frac{1}{2}\begin{vmatrix}1 & 1 & 1\\ x_1 & x & x_3\\ y_1 & y & y_3\end{vmatrix}$$

$$N_i(x,y)=\frac{1}{2A_e}\left(a_i+b_ix+c_iy\right)$$

$$a_i=x_jy_k-x_ky_j$$
$$b_i=y_j-y_k$$
$$c_i=x_k-x_j$$

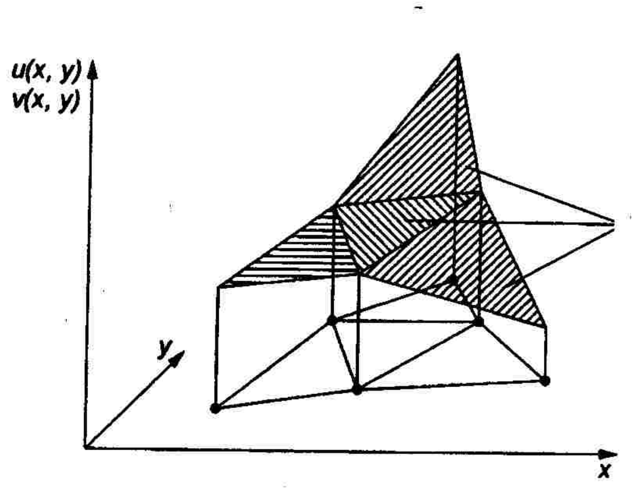
$$\{\varepsilon\}=\begin{Bmatrix}\varepsilon_x(x,y)\\ \varepsilon_y(x,y)\\ \gamma_{xy}(x,y)\end{Bmatrix}=\underset{3\times2}{[R]}\underset{2\times1}{\begin{Bmatrix}u_{(x,y)}\\ v_{(x,y)}\end{Bmatrix}}=\underset{3\times2}{[R]}\underset{2\times6}{\left[N_{(x,y)}\right]}\underset{6\times1}{\{q\}_e}$$

$$\{\varepsilon\}=\underset{3\times6}{[B]}\underset{6\times1}{\{q\}_e}$$



Strain- displacement matrix $[B]$:

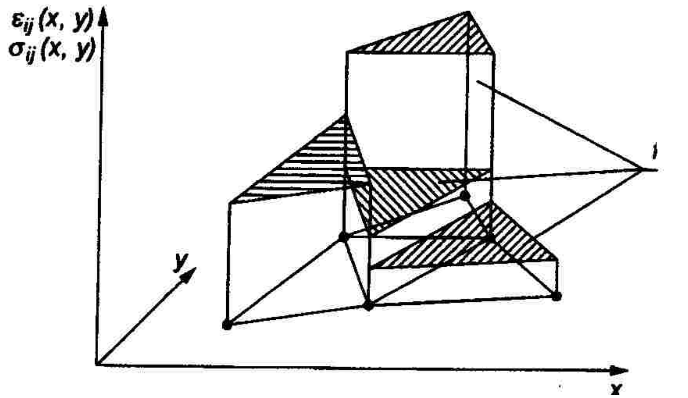
$$[B]=[R][N]=\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}$$



$$[B]=\frac{1}{2A_e}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

With constant coefficients for each finite element.

CST – constant strain triangle! - linear displacement field within elements and constant strains and stresses



$$\{\sigma\}=[D]\{\varepsilon\}$$

$$\{\sigma\}=[D][B]\{q\}_e$$

STRAIN ENERGY OF THE ELEMENT

$$U_e = h_e \int_{A_e} \frac{1}{2} \lfloor \varepsilon \rfloor \{\sigma\} dA_e = A_e h_e \frac{1}{2} \lfloor q \rfloor_e [B]^T [D][B]\{q\}_e$$

$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\} e$$

The stiffines matrix of the CST element $[k]_e$

$$[k]_e = \frac{1}{2} A_e h_e \underset{6\times 3}{[B]^T} \underset{3\times 3}{[D]} \underset{3\times 6}{[B]}$$

The strain energy of the entire model (N degrees of freedom)

$$U = \frac{1}{2} \lfloor q \rfloor [K] \{q\}$$

where $\{q\}$ is the total nodal displacement vector. $[K]$ matrix – symmetrical, semi-positive defined , singular

$$V = U - W_z = \frac{1}{2} \underset{1\times n}{\lfloor q \rfloor} \underset{n\times n}{[K]} \underset{n\times 1}{\{q\}} - \underset{1\times n}{\lfloor q \rfloor} \underset{n\times 1}{\{F\}} = \min!$$

Global nodal forces vector $\{F\}$ is assembled from the equivalent nodal forces of all elements

Minimum of V with respect to {q} → $[K]\{q\} = \{F\}$

Nodal forces of the Ω_e element equivalent to the body load $\lfloor X \rfloor$:

$$W_z^x = \int_{\Omega_e} \lfloor X \rfloor \{u\} d\Omega_e = \int_{\Omega_e} \lfloor X \rfloor [N] \{q\}_e d\Omega_e = \lfloor F^x \rfloor_e \{q\}_e ,$$

$$\lfloor F^x \rfloor_e = \int_{\Omega_e} \lfloor X \rfloor [N] d\Omega_e \qquad \left(\text{e.g. } F_1^X = \int_{\Omega_e} X_1(x,y) N_1(x,y) d\Omega_e \right)$$

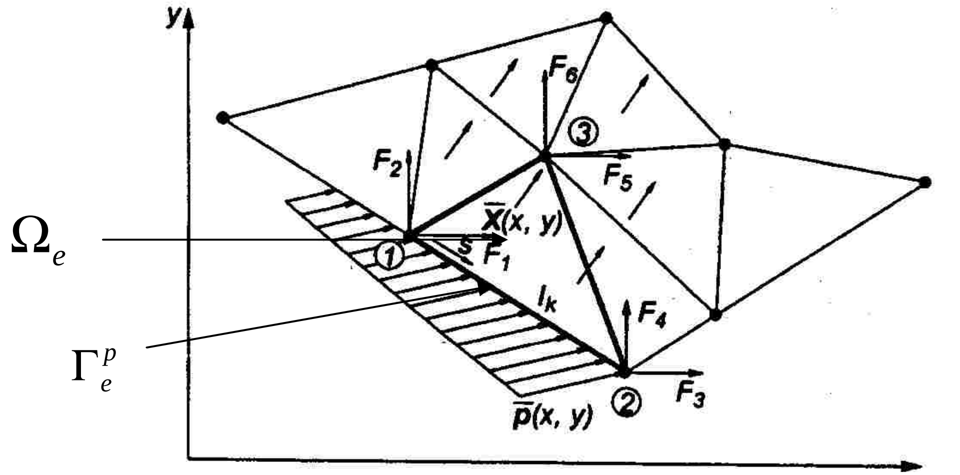
Nodal forces equivalent to the surface traction p acting on the edge Γ_e^p of the element Ω_e

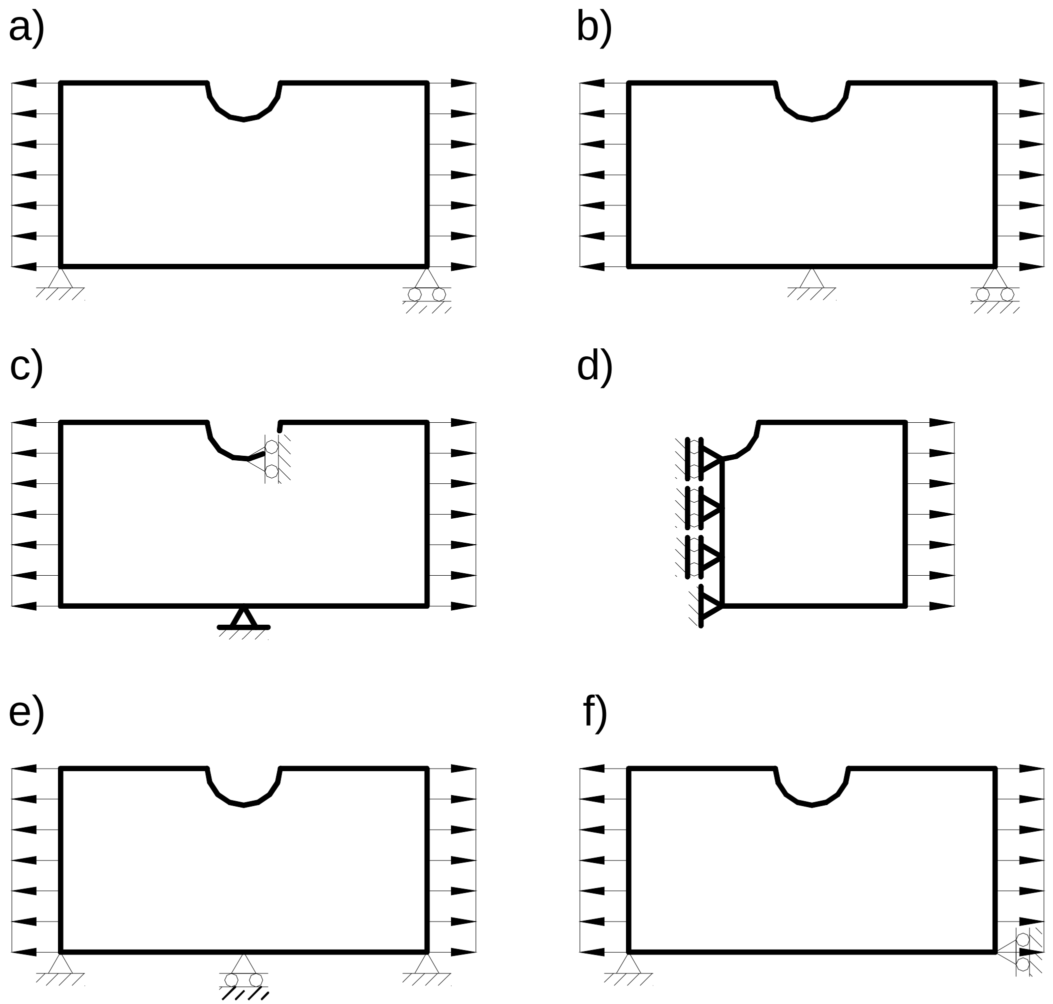
$$W_z^p = \int_{\Gamma_e^p} \lfloor p \rfloor \{u\} d\Gamma_e^p = \int_{\Gamma_e^p} \lfloor p \rfloor [N] \{q\}_e d\Gamma_e^p = \lfloor F^p \rfloor_e \{q\}_e ,$$

$$\lfloor F^p \rfloor_e = \int_{\Gamma_e^p} \lfloor p \rfloor [N] d\Gamma_e^p .$$

The total stiffnes matrix **K** is singular – the system of linear equations is modified by taking into account the current displacement boundary conditions.

$$\lfloor F \rfloor_e = \lfloor F_1, F_2, F_3, F_4, F_5, F_6 \rfloor = \lfloor F^X \rfloor_e + \lfloor F^p \rfloor_e$$





The 2D model of a tensioned plate (under external loads being in equilibruim). The correct and incorrect constraints (constrained rigid body motion,unconstrained deformation)

Finite element program:

Preprocessor
Information describing
- the geometry,
- the material properties ,
- the loads, the displacement boundary conditions.

Discretization of the model using the chosen type of finite elemets (e.g. CST)

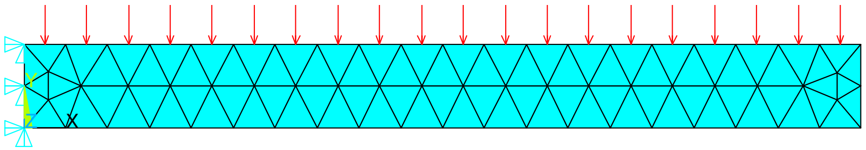
Processor
Assembling the stiffness matrix using the stiffness matrices of all finite elements
- Building the set of simultaneous equations with included boundary conditions (displacement b.c. and equivalent nodal forces)
- Solution of the set of equations – calculation of all nodal displacements

Calculation of strain and stress components within all finite elements

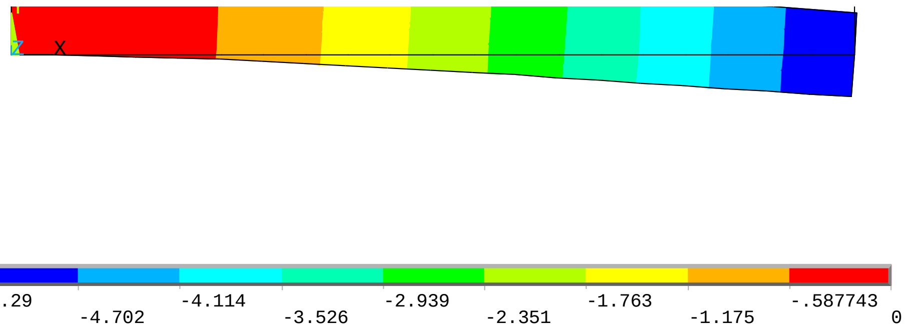
Postprocessor
Graphical presentation of the results (contour maps, isolines , isosurfaces, graphs, animations)
Listings, tables
User defined operations on the received results

RESULTS OBTAINED USING CST ELEMENTS - AVERAGING

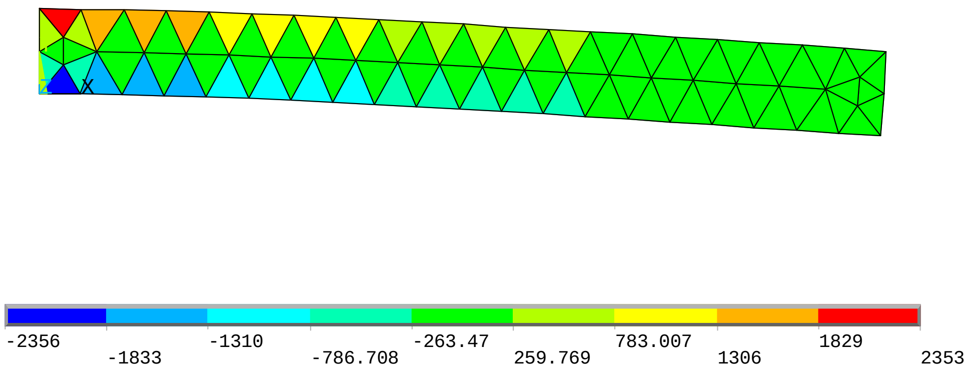
Example –2D FE model of the cantilever beam
Finite element mesh



Vertical displacement distribution

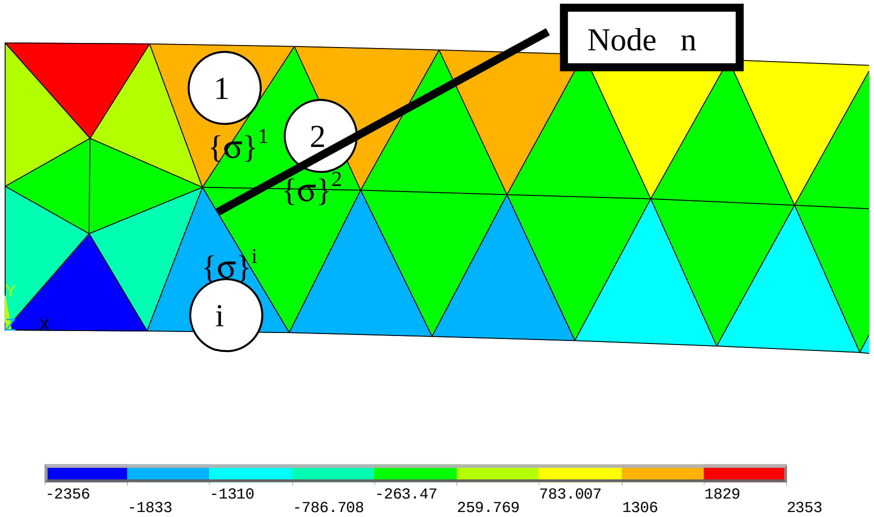


Bending stress (σ_x) distribution
(element solution)

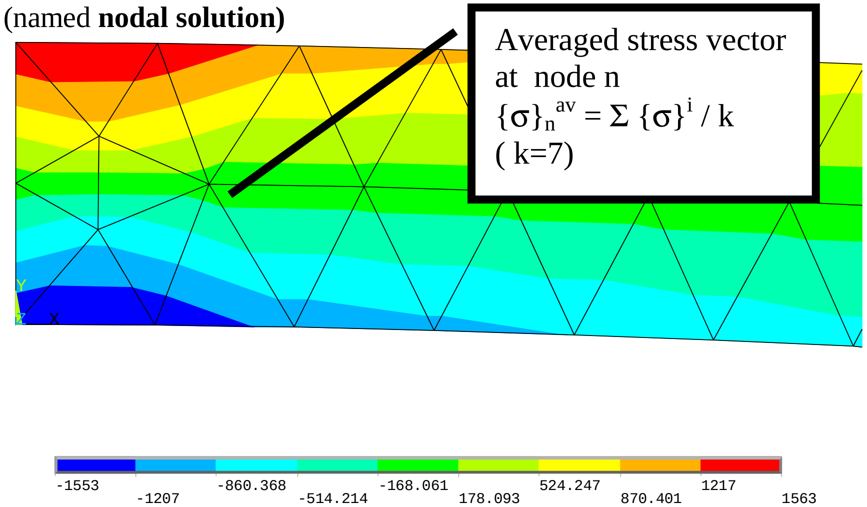


stress vectors in the CST elements

$$\{\sigma\}^1 \neq \{\sigma\}^2 \neq \{\sigma\}^3 \neq \ldots$$



Averaged presentation (named **nodal solution)**

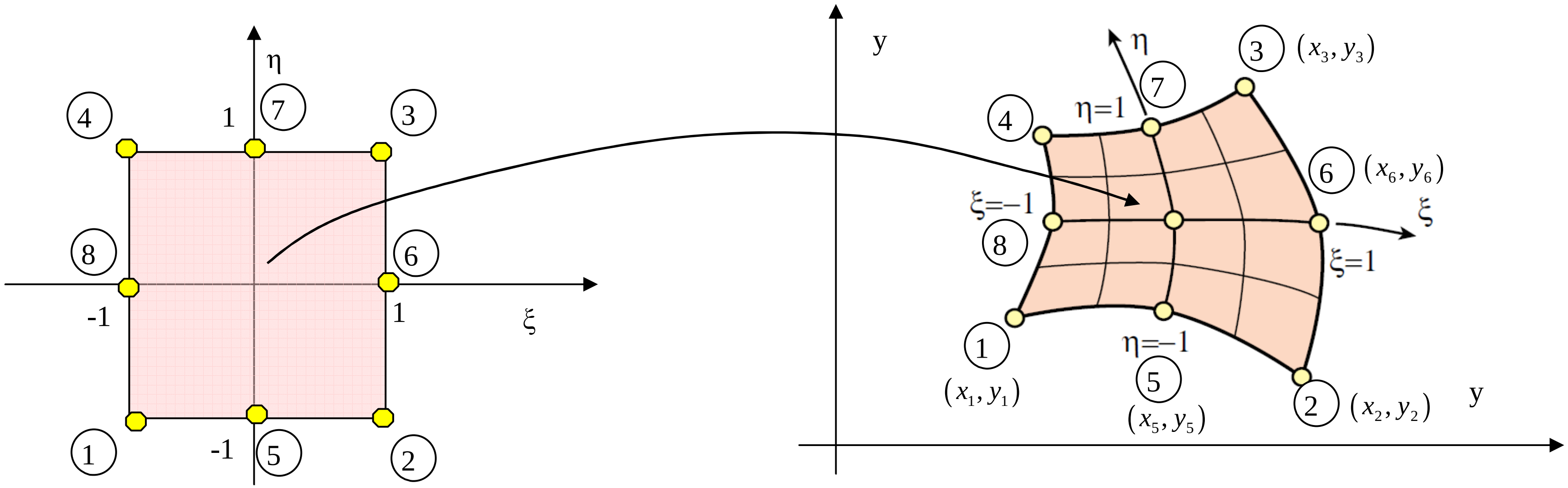


9. 8-NODE QUADRILATERAL ELEMENT. NUMERICAL INTEGRATION

The technique used for the formulation of the linear triangle can be formally extended to construct quadrilateral elements as well as higher order triangles. But it is connected with some difficulties:

1. The construction of shape functions satisfying consistency requirements for higher order elements with curved boundaries becomes increasingly difficult.
2. Computations of shape function derivatives to evaluate the strain-displacement matrix.
3. Integrals that appear in the expressions of the element stiffness matrix and consistent nodal force vector can no longer be carried out in closed form.

The 8-node element is defined by eight nodes having two degrees of freedom at each node: translations in the nodal x $(\boldsymbol{u})$ and y directions $(\boldsymbol{v})$. It provides more accurate results and can tolerate irregular shapes without much loss of accuracy. The 8-node are well suited to model curved boundaries.



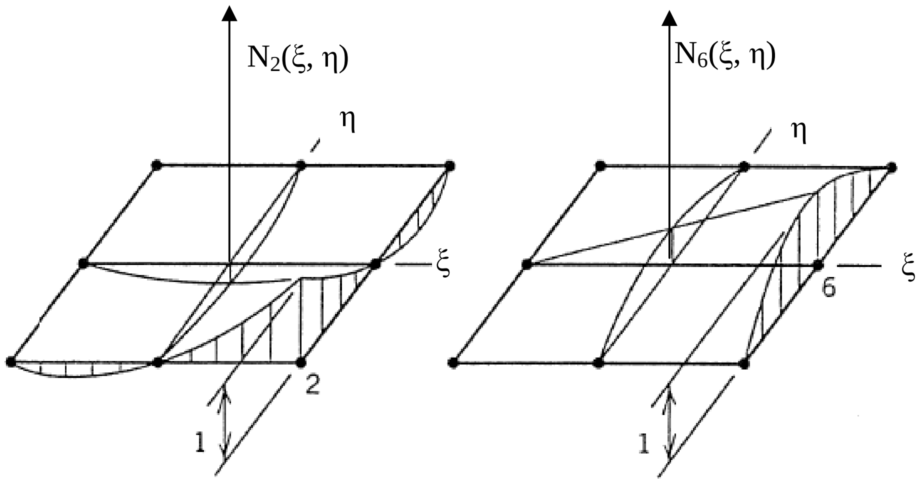
$$(\xi,\eta) \to (x, y)$$

$$(-1,-1) \to (x_1, y_1) \qquad (1,-1) \to (x_2, y_2) \qquad (1,1) \to (x_3, y_3) \qquad (-1,1) \to (x_4, y_4)$$

$$(0,-1) \to (x_5, y_5) \qquad (1,0) \to (x_6, y_6) \qquad (0,1) \to (x_7, y_7) \qquad (-1,0) \to (x_8, y_8)$$

$$x(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)\,x_i$$

$$y(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)\,y_i$$



$$N_1(\xi,\eta)=-\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \qquad N_5(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_2(\xi,\eta)=-\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \qquad N_6(\xi,\eta)=\frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_3(\xi,\eta)=-\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta) \qquad N_7(\xi,\eta)=\frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_4(\xi,\eta)=-\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \qquad N_8(\xi,\eta)=\frac{1}{2}(1-\xi)(1-\eta^2)$$

Shape functions N_2 and N_6

$$\begin{Bmatrix} x \\ y \end{Bmatrix}=\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & \dots & 0 & N_8 \end{bmatrix}\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_8 \\ y_8 \end{Bmatrix}_e \qquad \begin{Bmatrix} x \\ y \end{Bmatrix}=[N]\begin{Bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ x_3 \\ y_3 \\ \vdots \\ x_8 \\ y_8 \end{Bmatrix}=[N]\{xy\}$$

$$u(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)\,u_i \qquad v(\xi,\eta)=\sum_{i=1}^{8}N_i(\xi,\eta)\,v_i$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & 0 & \dots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 & N_5 & \dots & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}_e \qquad \begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{q\}_e$$

$$\underset{3\times1}{\{\varepsilon\}} = \underset{3\times2}{[R]}\underset{2\times1}{\{u\}} = \underset{3\times2}{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}} \underset{2\times16}{\left[N(\xi,\eta)\right]} \underset{16\times1}{\{q\}_e} = \underset{3\times16}{[B]}\underset{16\times1}{\{q\}}$$

$$[B] = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \dots & \frac{\partial N_8}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & \dots & 0 & \frac{\partial N_8}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \dots & \frac{\partial N_8}{\partial y} & \frac{\partial N_8}{\partial x} \end{bmatrix}$$

Partial derivatives of shape functions with respect to the Cartesian coordinates x and y are required for the strain and stress calculations. Since the shape functions are not directly functions of x and y but of the natural (local) coordinates ξ and η, the determination of Cartesian partial derivatives is not trivial.

We need the Jacobian of two-dimensional transformations that connect the differentials of $\{x, y\}$ to those of $\{\xi, \eta\}$ and vice-versa

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} \cdot x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\delta \xi} \cdot y_i \\ \sum_{i=1}^{8} \frac{\partial N_i}{\partial \eta} \cdot x_i & \sum_{i=1}^{8} \frac{\partial N_i}{\eta} \cdot y_i \end{bmatrix} = \left[J(\xi,\eta)\right]$$

Matrix **J** is called the *Jacobian matrix* of (x, y) with respect to (ξ, η), whereas $\mathbf{J}^{-1}$ is the Jacobian matrix of (ξ, η) with respect to (x, y). **J** and $\mathbf{J}^{-1}$ are often called the *Jacobian* and *inverse Jacobian*, respectively. The scalar symbol J means the determinant of **J**: $J = |\mathbf{J}| = \det \mathbf{J}$. Jacobians play a crucial role in differential geometry.

$$\frac{\partial N_i}{\partial x} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial x}$$
$$\frac{\partial N_i}{\partial y} = \frac{\partial N_i}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial N_i}{\partial \eta} \frac{\partial \eta}{\partial y}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \eta}{\partial x} \\ \dfrac{\partial \xi}{\partial y} & \dfrac{\partial \eta}{\partial y} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial N_i}{\partial \xi} \\ \dfrac{\partial N_i}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N_i}{\partial x} \\ \dfrac{\partial N_i}{\partial y} \end{Bmatrix}$$

$$\{\varepsilon\} = \left[B(\xi,\eta)\right]\{q\}_e$$

$$U_e = \int_{\Omega_e(x,y))} \frac{1}{2} \lfloor \varepsilon \rfloor \{\sigma\}\, dxdy = \frac{1}{2} \int_{\Omega_e(x,y)} \lfloor q \rfloor_e [B]^T [D][B]\{q\}_e\, dxdy$$

$$\int_{A(x,y)} f(x,y)\, dxdy = \int_{A(\xi,\eta)} f(\xi,\eta) \det[J]\, d\xi d\eta \qquad dxdy = \det[J]\, d\xi d\eta$$

$$U_e = 1/2 \lfloor q \rfloor_e \int_{\Omega_e(\xi,\eta)} \underset{16\times3}{\left[B(\xi,\eta)\right]^T} \underset{3\times3}{[D]} \underset{3\times16}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi d\eta\ \{q\}_e$$

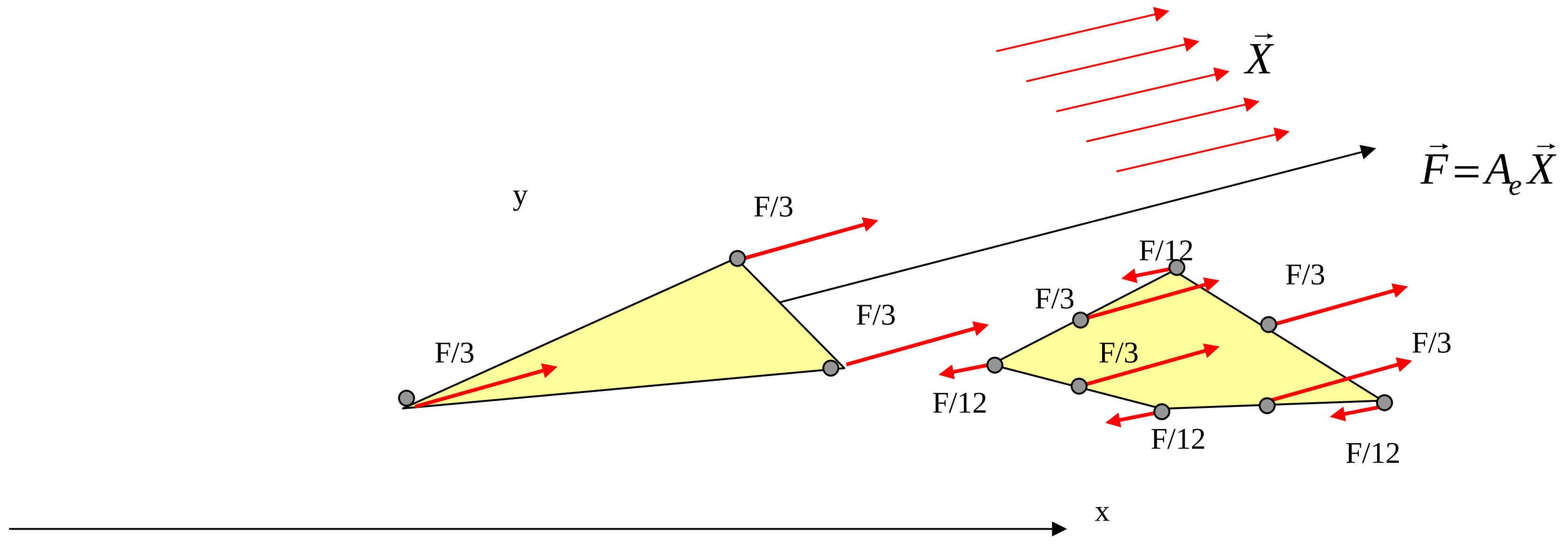
$$U_e = \frac{1}{2} \lfloor q \rfloor_e [k]_e \{q\}_e$$

$$[k]_e = \int_{\Omega_e(x,y)} [B]^T [D][B] dxdy = \int_{-1}^{1} \int_{-1}^{1} \underset{16x3}{\left[B(\xi,\eta)\right]^T} \underset{3\times3}{[D]} \underset{3\times16}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi d\eta \int \underset{16\times3}{\left[B(\xi,\eta)\right]}$$

Nodal forces of the Ω_e element equivalent to <u>the body load</u>:

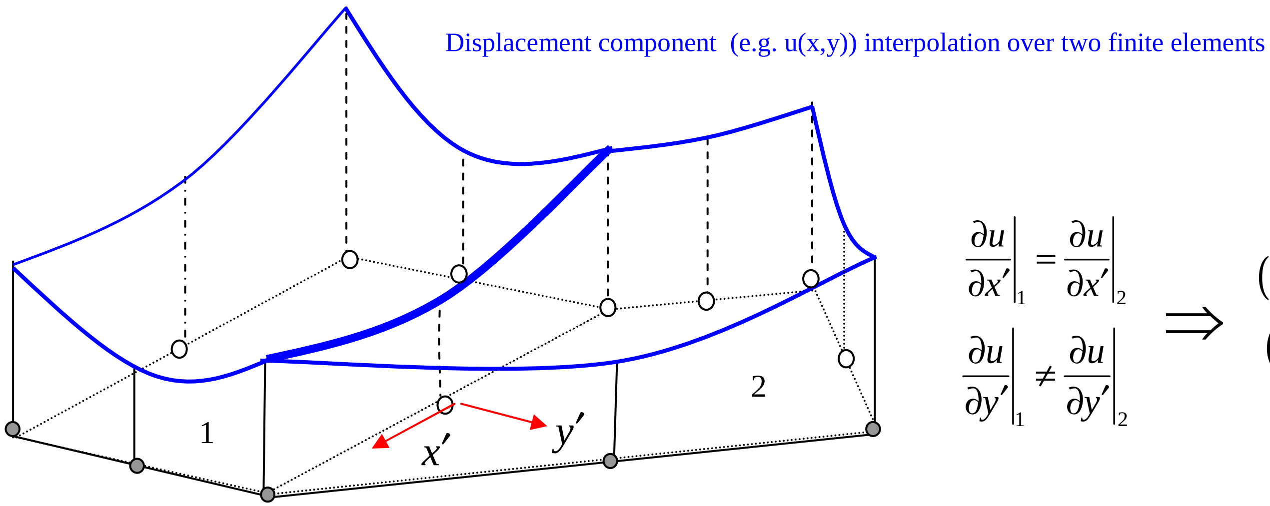
$$W_z^x = \int_{\Omega_e} \lfloor X \rfloor \{u\} d\Omega_e = \int_{\Omega_e} \lfloor X \rfloor [N] \{q\}_e \, d\Omega_e = \lfloor F^x \rfloor_e \{q\}_e ,$$

$$\lfloor F^x \rfloor_e = \int_{\Omega_e} \lfloor X \rfloor [N] d\Omega_e .$$



Work-equivalent nodal forces for uniform constant body load in the case of CST element and 8-node quadrilateral element

Finite element method results: continuous displacement field and discontinous stress field



$$\left.\frac{\partial u}{\partial x'}\right|_1 = \left.\frac{\partial u}{\partial x'}\right|_2 \qquad \left.\frac{\partial u}{\partial y'}\right|_1 \neq \left.\frac{\partial u}{\partial y'}\right|_2 \quad \Longrightarrow \quad \left(\varepsilon_{x'}\right)_1 = \left(\varepsilon_{x'}\right)_2 \qquad \left(\varepsilon_{y'}\right)_1 \neq \left(\varepsilon_{y'}\right)_2 \quad \Longrightarrow \left(\sigma_{ij}\right)_1 \neq \left(\sigma_{ij}\right)_2$$

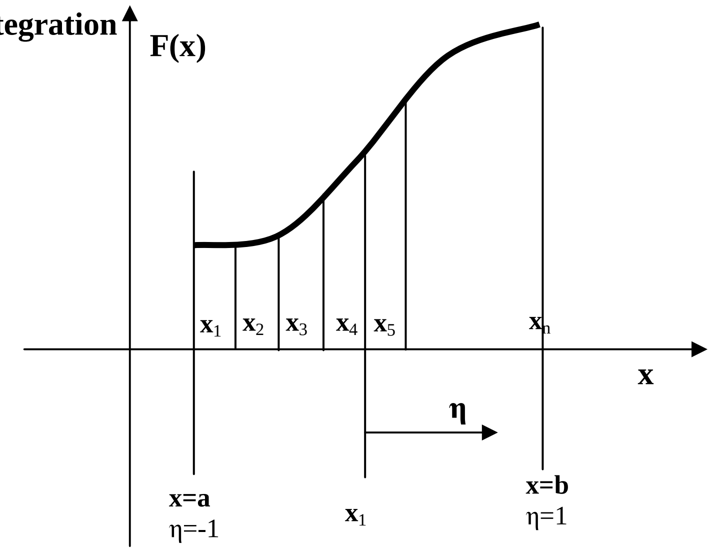
Numerical Gauss integration in FE algorihms

The use of numerical integration is essential for evaluating element integrals of isoparametric elements. The standard practice has been to use *Gauss integration*[1] because such rules use a minimal number of sample points to achieve a desired level of accuracy. This property is important for efficient element calculations because we shall see that at each sample point we must evaluate a matrix product.

$$[k]_e = \int\limits_{\Omega_e(\xi,\eta)} [B]^T [D][B]\,dxdy = \int\limits_{\Omega_e(x,y)} \underset{16\times3}{\left[B(\xi,\eta)\right]^T} \underset{3\times3}{[D]} \underset{3\times16}{\left[B(\xi,\eta)\right]} \det\left[J(\xi,\eta)\right] d\xi d\eta$$

The numerical intergration have to be also performed for finding the equivalent nodal forces.

One dimensional integration



In general:

$$\int_a^b F(x)\,dx = \sum_{i=1}^{n} \alpha_i F_i(x_i) + R_n$$

Introducing the new variable -1 ≤η≤ 1

$$x = \frac{(a+b)}{2} + \frac{b-a}{2}\cdot\eta \qquad dx = \frac{b-a}{2}\eta$$

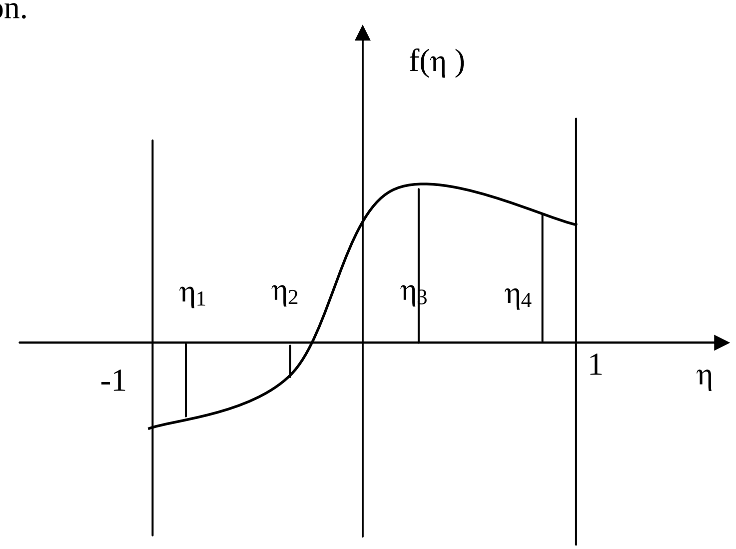
$$\int_a^b F(x)\,dx = \int_{-1}^{1} f(\eta)\frac{b-a}{2}d\eta = \frac{b-a}{2}\int_{-1}^{1} f(\eta)\,d\eta$$

The Gauss integration

$$\int_{-1}^{1} f(\eta)\,d\eta = \sum_{i=1}^{n} w_i f(\eta_i) + R_n \qquad R_n = 0\left(\frac{d^{2n} f}{d\eta^{2n}}\right)$$

Here $n \geq 1$ is the number of especially defined Gauss integration points, w_i are the integration weights, and ηi are sample-point abcissae in the interval [−1,1]. The use of the interval [−1,1] is no restriction, because an integral over another range, from *a* to *b* can be transformed to the standard interval via a simple linear transformation of the independent variable, as shown above. The values η_i and w_i are defined in such a way to aim for best accuracy. Indeed, if we assume a polynomial expression, it is easy to check that for *n* sampling a polynomial of degree $2n - 1$ can be exactly integrated .

Table below shows the positions and weighting coefficients for gaussian integration.



Abscissae and weight coefficients of the gaussian Quadrature

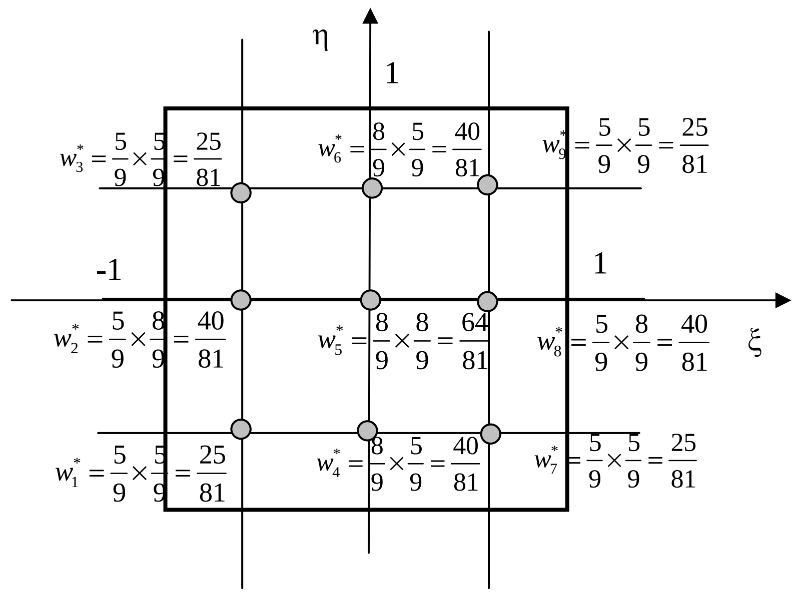
n	ς_i (i=1,n)	W_i (i=1,n)
1	0	2
2	$-1/\sqrt{3}$ $+1/\sqrt{3}$	1 1
3	$-\sqrt{0.6}$ 0 $+\sqrt{0.6}$	5/9 8/9 5/9
4	-0.861136311594953 -0.339981043584856 +0.339981043584856 +0.861136311594953	0.347854845137454 0.652145154862546 0.652145154862546 0.347854845137454
5		

Remarks: The sum of weighing coefficients is always 2
The integration gives the exact solution for polynomials of 2n-1 degree.

Numerical integration – rectangular region:

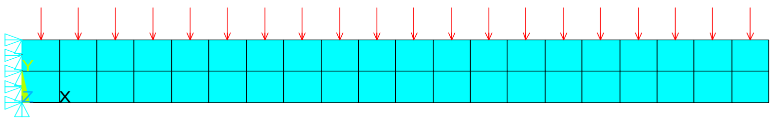
$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta \approx \int_{-1}^{1}\left(\sum_{i=1}^{n} f(\xi_i,\eta)\, w_i\right) d\eta \approx \sum_{j=1}^{n} w_j \sum_{i=1}^{n} w_i\, f(\xi_i,\eta_j) =$$

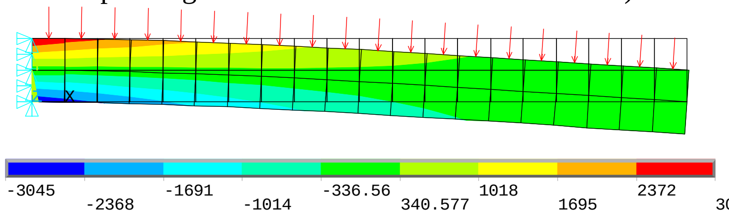
$$= \sum_{i=1}^{n}\sum_{i=1}^{n} w_i w_j\, f(\xi_i,\eta_j) = \sum_{k=1}^{m} w_k^*\ f(\xi_k,\eta_k)$$



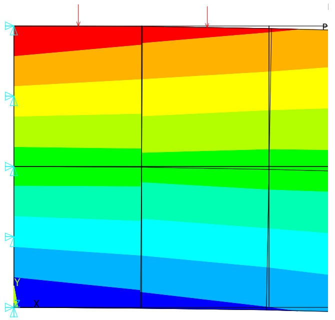
RESULTS OBTAINED USING 8-NODE ELEMENTS - AVERAGING

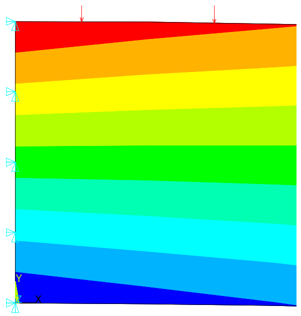
Example –2D FE model of the cantilever beam (compare to the results corresponding to discretization with CST elements)

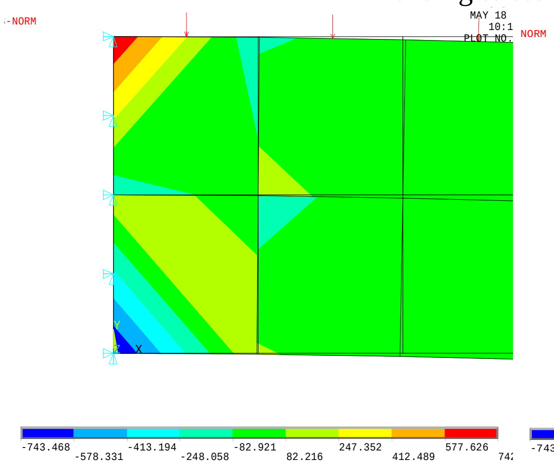


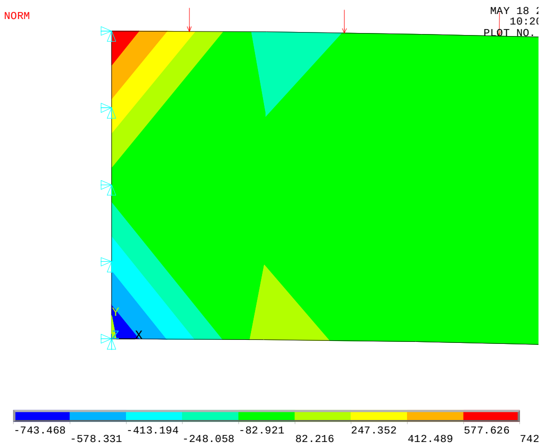


Bending stress σ_x distribution (element solution)









σ_x element and nodal solution σ_y element and nodal solution